SUPERVISOR TO ATTACH PROCESSING LABEL HERE

VCE Mathematical Methods
Units 3 & 4
Practice examination

Letter

STUDENT NUMBER

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book
- Formula sheet
- Answer sheet for multiple-choice questions
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- In questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1

(20 marks)

1 A transformation maps the point (x, y) to the point $(3x+4, 18y+5)$.

Under this transformation, the image of the graph of $y = x^2$ has the equation

A $y = 2(x-4)^2 - 5$

B $y = 2(x-4)^2 + 5$

C $y = 2(x-5)^2 + 4$

D $y = 2(x+4)^2 + 5$

E $y = 18(x-4)^2 + 5$

(1 mark)

2 Let $q : (-\infty, -2) \to \mathbb{R}$, where $q(x) = \log_e(x^2-4) - \log_e(2-x)$. The value of $q'(-5)$ is

A $-\frac{1}{5}$

B $-\frac{1}{7}$

C $-\frac{2}{3}$

D $\frac{1}{3}$

E $-\frac{1}{3}$

(1 mark)

3 If $\int f(x)\,dx = F(x)+c$ and $\int g(x)\,dx = G(x)+c$, then $\int (9f(x)-5g(x))\,dx$ is equal to

A $9G(x) - 5F(x) + c$

B $9F(x) - 5G(x) + c$

C $F(x) - G(x) + c$

D $5F(x) - 9G(x) + c$

E $9F(x) + 5G(x) + c$

(1 mark)

4 A bag has 9 red pens and 3 black pens. Two are drawn without replacement. The probability of one of each colour is

A $\frac{9}{44}$

B $\frac{13}{22}$

C $\frac{9}{22}$

D $\frac{3}{16}$

E $\frac{3}{8}$

(1 mark)

5 A differentiable function f satisfies $f'(x) = (-x-5)(x+2)$. The complete set of values of x for which $f'(x) > 0$ is

A $(-5, -2)$

B $(-2, \infty)$

C $(-\infty, -5) \cup (-2, \infty)$

D $(-\infty, -5)$

E $\mathbb{R}$

(1 mark)

6 The oblique asymptote of the graph of $f(x) = \frac{-4x^2-19x-17}{x+4}$ is

A $y = 3 - 4x$

B $y = -3x - 4$

C $y = -4x - 8$

D $y = -4x - 3$

E $x = -4$

(1 mark)

7 If $\int f(x)\,dx = F(x) + c$, then $\int f(3x+7)\,dx$ is equal to

A $\dfrac{1}{7}F(3x+7) + c$

B $\dfrac{1}{3}F(x) + c$

C $3F(3x+7) + c$

D $\dfrac{1}{3}F(3x+7) + c$

E $F(3x+7) + c$

(1 mark)

8 A box has 2 red marbles and 3 blue marbles. Two are drawn without replacement. The probability of the same colour is

A $\frac{2}{5}$

B $\frac{13}{25}$

C $\frac{1}{10}$

D $\frac{3}{5}$

E $\frac{8}{25}$

(1 mark)

9 The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} -x + 3(x-1)^2 - 4 & x \le 1, \\ 5x - 10 & x > 1. \end{cases}$$

At $x = 1$, the function is

A undefined at the join

B continuous and differentiable

C differentiable but not continuous

D continuous but not differentiable

E neither continuous nor differentiable

(1 mark)

10 For $f(x) = -3\tan(2x+6) - 1$, the two vertical asymptotes immediately adjacent to $x = -3$ are

A $x = 3 - \frac{\pi}{4}, \quad x = \frac{\pi}{4} + 3$

B $x=-\pi-3, \quad x=-3+\pi$

C $y=-3-\frac{\pi}{4}, \quad y=-3+\frac{\pi}{4}$

D $x=-3-\frac{\pi}{2}, \quad x=-3+\frac{\pi}{2}$

E $x=-3-\frac{\pi}{4}, \quad x=-3+\frac{\pi}{4}$

(1 mark)

11 A function g is continuous on $[0,8]$ and differentiable at $x=4$. Its average rate of change over the interval is 3 and $g'(4)=-1$. Therefore, on this interval, g must be

A constant

B strictly decreasing

C one-to-one

D strictly increasing

E many-to-one

(1 mark)

12 The function f is defined by

$$f(x)=\begin{cases}2-3x & x\leq 0\\ x+2 & x>0\end{cases}$$

The average value of f over the interval $[-3,1]$ is

A 7

B 2

C $\frac{11}{2}$

D 22

E $\frac{5}{2}$

(1 mark)

13 Two independent challenges have success probabilities $\frac{1}{5}$ and $\frac{4}{5}$. The probability of exactly one success is

A $\frac{1}{25}$

B $\frac{4}{25}$

C $\frac{17}{25}$

D $\frac{16}{25}$

E $\frac{21}{25}$

(1 mark)

14 The point $(7,-11)$ lies on the graph of $y=4f(x-4)-3$.

The corresponding point on the graph of $y=f(x)$ is

A $(3,-11)$

B $(11,-11)$

C $(3,-8)$

D $(11,-2)$

E $(3,-2)$

(1 mark)

15 A differentiable function f satisfies $f'(x) = -x(x-6)(x-4)$. The number of local maximum points of f is

A 3

B 2

C 4

D 1

E 0

(1 mark)

16 Which one of the following is not an antiderivative of $f(x) = -9x^2 - 1$?

A $-3x^3 - x - 4$

B $-3x^3 - x - 3$

C $-3x^3 - x + 2$

D $-3x^3 + x$

E $-3x^3 - x - 1$

(1 mark)

17 Let $f(x) = x^2 + 5$ and $g(x) = 2x - 4$.

The value of $f(g(-3))$ is

A -24

B 105

C 24

D 201

E 100

(1 mark)

18 A bag has 5 red tokens and 3 blue tokens. Three are drawn without replacement. The probability of red, blue, red, in that order, is

A $\frac{75}{512}$

B $\frac{15}{128}$

C $\frac{15}{28}$

D $\frac{25}{112}$

E $\frac{5}{28}$

(1 mark)

19 For $x > 4$, let $f(x) = \log_e\left(\frac{x+4}{x-4}\right)$. The gradient of the graph of f at $x = 16$ is equal to

A $-\frac{1}{60}$

B $-\frac{1}{30}$

C $-\frac{8}{15}$

D $\frac{1}{30}$

E $-\frac{1}{6}$

(1 mark)

20 The function $f : R \to R$ is defined by

$$f(x) = \begin{cases} ax+3 & x < 4 \\ x^2+3 & x \geq 4 \end{cases}$$

where a is a real constant.

If f is continuous at $x = 4$, then a is equal to

A -4

B 16

C 4

D 8

E 19

(1 mark)

Question 2

A nursery records the height of each seedling after one month. Let X be the height, in centimetres, with probability density function

$$f(x) = \begin{cases} kx^2(4-x) & 0 \leq x \leq 4, \\ 0 & \text{elsewhere,} \end{cases}$$

where k is a real constant.

a Show that $k = \frac{3}{64}$. (2 marks)

b Find the probability that the height is less than 1 centimetres. Give your answer correct to four decimal places. (2 marks)

c Find $E(X)$, the mean height. (2 marks) d Find the standard deviation of X. Give your answer correct to two decimal places.(2 marks)

e Find the median height, m, where $\Pr(X < m) = 0.5$. Give your answer correct to two decimal places. (2 marks)

Question 3

a The population of feral goats can be modelled by the rule $r(t) = 1500\sin\left(\frac{\pi t}{90}\right) + 2200$.

i State the initial population of feral goats. (1 mark)

ii State the minimum and maximum population of feral goats. (1 mark)

iii State the number of weeks between maximum populations of feral goats. (1 mark)

b One point of minimum dingo population, $(30, 500)$, and one point of maximum dingo population, $(120, 2300)$, are recorded. The population of dingoes can be modelled by the rule $f(t) = a\sin(b(t-75)) + 1400$.

i Show that $a = 900$ and $b = \frac{\pi}{90}$. (2 marks) ii Find the maximum combined population of dingoes and feral goats. Give your answer correct to the nearest whole number. (1 mark)

iii What is the number of weeks between the periods when the combined population of dingoes and feral goats is a maximum? (1 mark)

c The population of dingoes is better modelled by the transformation of $y = \sin(t)$ under Q given by

$$Q : R^2 \to R^2,\ Q\left(\begin{bmatrix} t \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{100}{\pi} & 0 \\ 0 & 900 \end{bmatrix} \begin{bmatrix} t \\ y \end{bmatrix} + \begin{bmatrix} 75 \\ 1400 \end{bmatrix}.$$

Find the average population during the first 300 weeks for the combined population of dingoes and feral goats, where the population of dingoes is modelled by the transformation of $y = \sin(t)$ under Q. Give your answer correct to the nearest whole number. (4 marks)

d Over a longer period of time, it is found that the increase and decrease in the population of feral goats gets smaller and smaller. The population of feral goats over a longer period of time can be modelled by the rule

$$s(t) = 1500e^{-0.002t} \sin\left(\frac{\pi t}{90}\right) + 2200, \quad \text{for all } t \geq 0.$$

i Find the average rate of change between the first two times when the population of feral goats is at a maximum. Give your answer correct to one decimal place. (2 marks)

ii Find the time, where $t > 50$, in weeks, when the rate of change of the feral goat population is at its greatest positive value. Give your answer correct to the nearest whole number. (2 marks) iii Over time, the feral goat population approaches a particular value. State this value. (1 mark)

Question 4

A cannery fills tins of tomatoes on one filling line. The mass of a randomly chosen tin from the filling line is a normally distributed random variable X, with a mean of 750 grams and a standard deviation of 12 grams.

a Find the probability that a randomly chosen tin has a mass greater than 765 grams. Give your answer correct to four decimal places. (1 mark)

b Find the probability that a randomly chosen tin has a mass greater than 765 grams, given that its mass is greater than 740 grams. Give your answer correct to four decimal places. (2 marks)

c Find the mass m for which $\Pr(X < m) = 0.25$. Give your answer in grams correct to one decimal place. (2 marks)

d A random sample of 100 tins is taken from the filling line. Let $\hat{P}$ be the random variable representing the proportion of tins in the sample that are underweight. Is $\hat{P}$ discrete or continuous? Justify your answer. (1 mark) e In that sample, 16 of the tins are underweight. Find an approximate 95% confidence interval for the proportion of underweight tins produced by the filling line. Give your answer correct to three decimal places. (2 marks)

f How many tins would need to be sampled to halve the width of the confidence interval found above? (1 mark)

g A crate holds 8 tins chosen at random from the filling line. Using the sample proportion above as the probability that a tin is underweight, find the probability that at least 3 of them are underweight. Give your answer correct to four decimal places. (3 marks)

Question 5

A horse is given a single dose of a sedative. The concentration of the drug in the bloodstream of the horse, in milligrams per litre, is modelled by

$$c(t) = 8te^{-0.4t}, \quad t \geq 0,$$

where t is the time in hours after the dose is given.

a State the concentration of the drug in the bloodstream of the horse when $t = 0$. (1 mark)

b Find the time at which the concentration is greatest, and find that greatest concentration. Give both answers correct to two decimal places. (3 marks) c Find the average concentration over the first 8 hours. Give your answer correct to two decimal places. (3 marks)

d Find the rate of change of the concentration when $t = 4$. Give your answer correct to two decimal places. (2 marks)

e After the concentration has peaked it falls. Find the time at which the concentration next falls to 4 milligrams per litre. Give your answer correct to two decimal places. (2 marks)

f State the concentration that the model approaches over time. (1 mark)

Question 6

A ferry runs to a small island each day. The depth of water at the jetty is modelled by

$$h(t) = 8 - 4\cos\left(\frac{\pi t}{6}\right),$$

where h is the depth in metres and t is the time in hours after low tide.

a State the maximum and minimum depth of water at the jetty. (1 mark) b State the period of the tide, in hours. (1 mark)

c Find the depth of water 4 hours after low tide. Give your answer in metres correct to two decimal places. (2 marks)

d The ferry needs a depth of at least 9 metres to enter the jetty. Find the first time after $t = 0$ at which it can enter. Give your answer in hours correct to two decimal places. (3 marks)

e Find the length of time, in each tidal cycle, for which the depth is at least 9 metres. Give your answer in hours correct to two decimal places. (3 marks)

Solutions

1. (1) **B.** Putting $X = 3x+4$ and $Y = 18y+5$ and substituting into $y = x^2$ gives $y = 2(x-4)^2+5$. The horizontal factor enters squared, so the coefficient is $18 \div 9 = 2$.
   (2) **E.** On the stated domain, $q(x) = \log_e(-(x+2))$, so $q'(x) = \frac{1}{x+2}$. Substitution gives $-\frac{1}{3}$.
   (3) **B.** Integration is linear, so each term keeps its own coefficient and its own sign, giving $9F(x) - 5G(x) + c$.
   (4) **C.** Either red then black or black then red can occur, so the probability is $\dfrac{9}{12}\dfrac{3}{11} + \dfrac{3}{12}\dfrac{9}{11} = \frac{9}{22}$.
   (5) **A.** The derivative has zeros at $x = -5$ and $x = -2$. Its sign is positive between the two roots and negative outside them, so the required set is $(-5, -2)$.
   (6) **D.** Polynomial division gives quotient - 4 x - 3 and remainder -5. The remainder divided by $x+4$ approaches zero as x becomes large in magnitude, so the oblique asymptote is $y = -4x-3$.
   (7) **D.** Differentiating $F(3x+7)$ brings out a factor of 3 by the chain rule, so the antiderivative carries $\dfrac{1}{3}$ to cancel it. The translation has no effect on the factor.

(8) **A.** Add the two disjoint paths: $\Pr(RR)+\Pr(BB)=\frac{2}{5}\frac{1}{4}+\frac{3}{5}\frac{2}{4}=\frac{2}{5}$.

(9) **D.** Both branches approach the value -5 at $x=1$, so f is continuous. Their one-sided gradients are -1 and 5, so f is not differentiable there.

(10) **E.** The adjacent asymptotes of $y=\tan(X)$ occur at $X=\pm\frac{\pi}{2}$. Solving $2x+6=\pm\frac{\pi}{2}$ gives $x=-3-\frac{\pi}{4}$ and $x=-3+\frac{\pi}{4}$. The vertical dilation and translation do not change these x-coordinates.

(11) **E.** The positive average rate gives $g(8)>g(0)$, but the negative derivative shows that g decreases locally at $x=4$. A continuous one-to-one function on an interval must be strictly monotonic, so g must be many-to-one.

(12) **C.** Integrate each piece over its own part of the interval, giving 22 altogether, then divide by the width 4.

(13) **C.** Exactly one success occurs on either of two disjoint paths. Thus $p_1(1-p_2)+(1-p_1)p_2=\frac{1}{25}+\frac{16}{25}=\frac{17}{25}$.

(14) **E.** Undo the steps in the opposite order: subtract -3 from the y-coordinate, divide by 4, and translate the x-coordinate back by -4, which gives $(3,-2)$.

(15) **B.** A local maximum occurs where f' changes from positive to negative. The three distinct simple zeros make the sign alternate, giving 2 positive-to-negative sign changes.

(16) **D.** Antiderivatives of the same function differ only by a constant. Every option but the first is $-3x^3-x$ plus a number; the first differs by a term in x, so differentiating it does not give f.

(17) **B.** $g(-3)=-10$, and applying f to that gives 105.

(18) **E.** Along the red-blue-red path, both the total and the available red count decrease: $\frac{5}{8}\frac{3}{7}\frac{4}{6}=\frac{5}{28}$.

(19) **B.** Differentiating the two logarithms gives $f'(x)=\frac{1}{x+4}-\frac{1}{x-4}=-\frac{8}{x^2-16}$. At $x=16$ this is $-\frac{1}{30}$.

(20) **C.** Continuity at the join needs the two branches to take the same value there: $a(4)+3=19$, so $a=4$. The gradients are not required to agree.

2. (a) Using the required working, the answer is

$$\int_0^4 kx^2(4-x)\,dx=1,\ k\left[\frac{4x^3}{3}-\frac{x^4}{4}\right]_0^4=k\times\frac{256}{12}=1,\ k=\frac{3}{64}$$

.

(b) Using the required working, the answer is $\Pr(X<1)=\int_0^1\frac{3}{64}x^2(4-x)\,dx,\ \approx 0.0508$.

(c) Using the required working, the answer is $E(X)=\int_0^4 x\times\frac{3}{64}x^2(4-x)\,dx,\ E(X)=\frac{12}{5}$ centimetres.

(d) Using the required working, the answer is

$$E\left(X^2\right)=\int_0^4 x^2\times\frac{3}{64}x^2(4-x)\,dx=\frac{32}{5}$$

, sd$(X)=\sqrt{E\left(X^2\right)-[E(X)]^2}=\frac{4}{5},\ \approx 0.80$ centimetres.

(e) Using the required working, the answer is $\int_0^m\frac{3}{64}x^2(4-x)\,dx=0.5,\ m\approx 2.46$ centimetres.

3. (a) (i) Using the required working, the answer is $r(0)=2200$.

(ii) Using the required working, the answer is minimum$=700$, maximum$=3700$.

(iii) Using the required working, the answer is period$=\frac{2\pi}{\frac{\pi}{90}}=180$ weeks.

(b) (i) $a=\frac{2300-500}{2}=900$, The maximum is half a period after the minimum, so the period is $2(120-30)=180,\ b=\frac{2\pi}{180}=\frac{\pi}{90}$

(ii) Using the required working, the answer is Both populations have period 180, so the sum is a single sinusoid, amplitude$= 849.55$, maximum$= 3600 + 849.55 \approx 4450$.

(iii) The sum has the same period as each population, 180 weeks

(c) Using the required working, the answer is Under Q the image of $y = \sin(t)$ is $y = 900\sin\left(\frac{\pi(t-75)}{100}\right) + 1400$, average

$$= \frac{1}{300}\int_0^{300}\left(1500\sin\left(\frac{\pi t}{90}\right) + 2200 + 900\sin\left(\frac{\pi(t-75)}{100}\right) + 1400\right)dt,\ \approx 3680$$

.

(d) (i) Use the average-rate formula $\frac{f(b)-f(a)}{b-a}$ over the given interval. This gives $s'(t) = 0$ where

$$\tan\left(\frac{\pi}{90}t\right) = \frac{\frac{\pi}{90}}{0.002},\ t_1 \approx 43.36,\quad t_2 \approx 223.36$$

, average rate

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \approx -2.3 \text{ per week}$$

.

(ii) $s''(t) = 0$ where

$$\tan\left(\frac{\pi}{90}t\right) = \frac{2 \times 0.002 \times \frac{\pi}{90}}{(0.002)^2 - \left(\frac{\pi}{90}\right)^2}$$

, The decay makes each later positive peak smaller, so the first one after $t = 50$ is the greatest, $t \approx 177$ weeks

(iii) Using the required working, the answer is $e^{-0.002t} \to 0$ as $t \to \infty,\ s(t) \to 2200$.

4. (a) Using the required working, the answer is $\Pr(X > 765) = 0.1056$.

(b) Using the required working, the answer is

$$\Pr(X > 765 \mid X > 740) = \frac{\Pr(X > 765)}{\Pr(X > 740)},\ = \frac{0.1056}{0.7977} = 0.1324$$

.

(c) Using the required working, the answer is $\Pr(X < m) = 0.25,\ m = 741.9$ grams.

(d) Using the required working, the answer is Discrete, $\hat{P} = \frac{X}{100}$, where X counts the underweight tins, so $\hat{P}$ takes only the values $0, \frac{1}{100}, \ldots, 1$.

(e) Using the required working, the answer is

$$\hat{p} = \frac{16}{100} = 0.160,\ \hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{100}},\ (0.088, 0.232)$$

.

(f) The width is proportional to $\frac{1}{\sqrt{n}}$, so halving it needs four times the sample, $n = 4 \times 100 = 400$

(g) Using the required working, the answer is $X \sim$Bi$(8, 0.160)$, $\Pr(X = 3) = 0.0959,\ \Pr(X \geq 3) = 0.1226$.

5. (a) Using the required working, the answer is $c(0) = 0$ milligrams per litre.

(b) Using the required working, the answer is $c'(t) = 8e^{-0.4t}(1 - 0.4t),\ c'(t) = 0$ when $t = \frac{1}{0.4},\ t \approx 2.50$ hours, $c(2.50) \approx 7.36$ milligrams per litre.

(c) Using the required working, the answer is average$= \frac{1}{8}\int_0^8 8te^{-0.4t}\,dt,\ \approx 5.18$ milligrams per litre.

(d) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $c'(4) \approx -0.97$ milligrams per litre per hour.

(e) Using the required working, the answer is $8te^{-0.4t} = 4,\ t \approx 6.36$ hours.

(f) Using the required working, the answer is $e^{-0.4t}$ decays faster than t grows, $c(t) \to 0$ milligrams per litre.

6. (a) Using the required working, the answer is maximum$= 12$ m, minimum$= 4$ m.

   (b) Using the required working, the answer is period$= 12$ hours.

   (c) Using the required working, the answer is $h(4) \approx 10.00$ m.

   (d) Using the required working, the answer is $\cos\left(\frac{\pi t}{6}\right) = \frac{8-9}{4},\ t \approx 3.48$ hours.

   (e) The depth is above the clearance between $t \approx 3.48$ and $t \approx 8.52$, time≈ 5.03 hours