SUPERVISOR TO ATTACH PROCESSING LABEL HERE

VCE Mathematical Methods
Units 3 & 4
Practice examination

Letter

STUDENT NUMBER

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book
- Formula sheet
- Answer sheet for multiple-choice questions
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- In questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1

(20 marks)

1 The point $(-2,5)$ lies on the graph of $y=f(x)$.
The graph of $y=f(x)$ is transformed to the graph of $y=-2f(x+2)+5$.
The coordinates of the image of that point are

A $(-4,10)$
B $(-4,-5)$
C $(-4,-20)$
D $(0,-5)$
E $(2,10)$

(1 mark)

2 For $-2<x<2$, the relation $e^{4y}=\frac{2+x}{2-x}$ defines y as a function of x. The value of $\frac{dy}{dx}$ at $x=0$ is

A 1
B $-\frac{1}{4}$
C 4
D $\frac{1}{4}$
E $\frac{1}{8}$

(1 mark)

3 If $\int_3^9 f(x)\,dx=-3$, then $\int_{-1}^1 f(3(x+2))\,dx$ is equal to

A -9
B $-\frac{1}{2}$
C -2
D -1
E -3

(1 mark)

4 Bag A has 2 red and 4 blue counters; Bag B has 4 red and 2 blue counters. A random counter is moved from A to B, then one is drawn from B. The probability it is red is

A $\frac{5}{21}$
B $\frac{4}{7}$
C $\frac{13}{21}$
D $\frac{2}{3}$
E $\frac{5}{7}$

(1 mark)

5 Let $g:(-\infty,0)\to\mathbb{R}$, where $g(x)=x\log_e(-4x)$. The x-coordinate of the stationary point of g is

A $-\frac{1}{e}$
B $-\frac{e}{4}$
C $-\frac{1}{4e}$
D $\frac{1}{4e}$

E $-\frac{1}{4}$

(1 mark)

6 The graph of $y = f(x)$ has asymptotes $x = -5$ and $y = -1$. Define $g(x) = 2f(3x-9)-2$. The asymptotes of the graph of $y = g(x)$ are

A $x = \frac{4}{3}, \quad y = -4$

B $x = \frac{14}{3}, \quad y = -4$

C $x = \frac{4}{3}, \quad y = -6$

D $x = -2, \quad y = -4$

E $x = \frac{4}{3}, \quad y = -3$

(1 mark)

7 If $\int_0^4 f(x)\,dx = -10$ and $\int_0^2 f(x)\,dx = -9$, then $\int_2^4 f(x)\,dx$ is equal to

A -19

B 1

C -9

D -1

E -10

(1 mark)

8 Let A and B be events such that $\Pr(A) = \frac{1}{4}$, $\Pr(B \mid A) = \frac{1}{5}$ and $\Pr(B \mid A') = n$. If A and B are independent, then n is equal to

A $\frac{4}{5}$

B $\frac{3}{4}$

C $\frac{1}{4}$

D $\frac{1}{20}$

E $\frac{1}{5}$

(1 mark)

9 Let $f : [-3,\infty) \to [4,\infty)$, where $f(x) = (x+3)^2+4$, and let $g = f^{-1}$. The value of $g'(29)$ is

A 10

B $\frac{1}{5}$

C $\frac{1}{10}$

D $\frac{5}{2}$

E $-\frac{1}{10}$

(1 mark)

10 The function with rule $f(x) = \dfrac{x^2-x-20}{x+1}$ is not continuous at

A $x = 4$

B $x = 0$

C $x = 1$

D $x = -4$

E $x = -1$

(1 mark)

11 A curve is modelled by $f(x) = 15\cos\left(\frac{\pi x}{100}\right) + 40$ for $0 \le x \le 100$. The gradient of the curve at $x = 50$ is equal to

A 0

B $-\frac{3\pi}{20}$

C $\frac{3\pi}{20}$

D $-\frac{3\pi}{40}$

E $-\frac{3}{20}$

(1 mark)

12 The value of $\int_{-3}^{3} \left(-3x^3 + 6x + 6\right) dx$ is

A 12

B 0

C 18

D 36

E -36

(1 mark)

13 The binomial random variable X has $n = 20$ and $p = \frac{7}{8}$. If $\Pr(X \ge 17 \mid X \ge a) = 0.7718$, correct to four decimal places, then a is equal to

A 6

B 13

C 3

D 14

E 15

(1 mark)

14 The graph of $y = f(x)$ is reflected in the x-axis and then translated 2 units in the negative direction of the y-axis.

The rule of the image is

A $y = f(-x) - 2$

B $y = -f(-x) - 2$

C $y = -f(x) - 2$

D $y = -f(x) + 2$

E $y = -f(x-2)$

(1 mark)

15 The one-to-one function f is differentiable, with $f(3) = \frac{\sqrt{2}}{2}$ and $f'(3) = 5$. Near $x = \frac{\pi}{12}$, the relation $f(y) = \sin(3x)$ defines y as a function of x. The value of $\frac{dy}{dx}$ at this point is

A $\frac{3\sqrt{2}}{10}$

B $\frac{3}{5}$

C $-\frac{3\sqrt{2}}{10}$

D $\frac{\sqrt{2}}{10}$

E $\frac{15\sqrt{2}}{2}$

(1 mark)

16 A function f is differentiable everywhere, with $f(-2) = -1$ and $f(1) = -7$.

The value of $\int_{-2}^{1} f'(x)\,dx$ is

A 6

B -2

C -6

D -7

E -8

(1 mark)

17 Which one of the following functions is not continuous over the interval $x \in [1,6]$?

A $\sqrt{x+1}$

B $x^2 - 4x - 1$

C $\sin(2x)$

D $\frac{1}{x-2}$

E $\frac{1}{x-8}$

(1 mark)

18 The discrete random variable X has the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	a	$4a$	$7a$	$10a$

The mean of X is

A $\frac{1}{22}$

B $\frac{3}{2}$

C 3

D 1

E $\frac{24}{11}$

(1 mark)

19 The largest value of a such that $f : (-\infty, a] \to \mathbb{R}$, where $f(x) = x^2 + 10x + 29$, is one-to-one is

A 4

B -6

C -5

D -4

E 5

(1 mark)

20 Let $f(x) = -2x - 4$ and $g(x) = -2x + 5$.

The inverse of the function with rule $f(g(x))$ has the rule

A $\frac{x}{4} + \frac{7}{2}$

B $\frac{x}{4} - \frac{13}{4}$

C $4x - 14$

D $\dfrac{1}{4x - 14}$

E $\frac{x}{4} - \frac{7}{2}$

(1 mark)

Question 2

A sheep is given a single dose of a wormer. The concentration of the drug in the bloodstream of the sheep, in milligrams per litre, is modelled by

$$c(t) = 5te^{-0.3t}, \quad t \geq 0,$$

where t is the time in hours after the dose is given.

a State the concentration of the drug in the bloodstream of the sheep when $t = 0$. (1 mark)

b Find the time at which the concentration is greatest, and find that greatest concentration. Give both answers correct to two decimal places. (3 marks)

c Find the average concentration over the first 6 hours. Give your answer correct to two decimal places. (3 marks)

d Find the rate of change of the concentration when $t = 4$. Give your answer correct to two decimal places. (2 marks) e After the concentration has peaked it falls. Find the time at which the concentration next falls to 3 milligrams per litre. Give your answer correct to two decimal places. (2 marks)

f State the concentration that the model approaches over time. (1 mark)

Question 3

A farmer is digging an irrigation channel across a paddock. The cross-section of the channel is modelled by

$$d(x) = 2 - \frac{x^2}{4},$$

where d is the depth below ground level in metres and x is the horizontal distance in metres from the centre of the channel, for $d(x) \geq 0$.

a Find the width of the channel at ground level. Give your answer in metres correct to two decimal places. (2 marks)

b State the greatest depth of the channel. (1 mark)

c Find the area of the cross-section of the channel. Give your answer in square metres correct to two decimal places. (3 marks) d Find the average depth of the channel across its width. Give your answer in metres correct to two decimal places. (2 marks)

e The channel is 40 metres long and has the same cross-section along its whole length. Find the volume of soil removed to dig it. Give your answer in cubic metres correct to two decimal places. (2 marks)

Question 4

A craft studio has a rectangular sheet of thin board measuring 40 cm by 16 cm. A square of side x cm is cut from each corner and the four flaps are folded up to form an open box, so that the box has volume

$$V(x) = x(40-2x)(16-2x),$$

where V is measured in cubic centimetres.

a Show that $V(x) = 4x^3 - 112x^2 + 640x$. (2 marks)

b State the values of x for which a box can be made. (1 mark)

c Find $V'(x)$. (1 mark)

d Find the value of x that gives the greatest volume. Give your answer in centimetres correct to two decimal places. (3 marks)

e Find the greatest volume of the box. Give your answer in cubic centimetres correct to two decimal places. (2 marks)

f Find the values of x for which the volume of the box is at least 750 cubic centimetres. Give your answers in centimetres correct to two decimal places. (3 marks)

Question 5

A council records how long each call to its help line lasts. Let X be the call length, in minutes, with probability density function

$$f(x) = \begin{cases} kx^2(5-x) & 0 \le x \le 5, \\ 0 & \text{elsewhere,} \end{cases}$$

where k is a real constant.

a Show that $k = \frac{12}{625}$. (2 marks)

b Find the probability that the call length is less than 2 minutes. Give your answer correct to four decimal places. (2 marks) c Find $E(X)$, the mean call length. (2 marks)

d Find the standard deviation of X. Give your answer correct to two decimal places. (2 marks)

e Find the median call length, m, where $\Pr(X < m) = 0.5$. Give your answer correct to two decimal places. (2 marks)

Question 6

a The population of field mice can be modelled by the rule $r(t) = 1300\sin\left(\frac{\pi t}{70}\right) + 2200$.

i State the initial population of field mice. (1 mark)

ii State the minimum and maximum population of field mice. (1 mark)

iii State the number of weeks between maximum populations of field mice. (1 mark)

b One point of minimum feral cat population, $(10, 500)$, and one point of maximum feral cat population, $(80, 2300)$, are recorded. The population of feral cats can be modelled by the rule $f(t) = a\sin(b(t-45)) + 1400$.

i Show that $a = 900$ and $b = \frac{\pi}{70}$. (2 marks)

ii Find the maximum combined population of feral cats and field mice. Give your answer correct to the nearest whole number. (1 mark)

iii What is the number of weeks between the periods when the combined population of feral cats and field mice is a maximum? (1 mark)

c The population of feral cats is better modelled by the transformation of $y = \sin(t)$ under Q given by

$$Q : R^2 \to R^2,\ Q\left(\begin{bmatrix} t \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{90}{\pi} & 0 \\ 0 & 900 \end{bmatrix}\begin{bmatrix} t \\ y \end{bmatrix} + \begin{bmatrix} 45 \\ 1400 \end{bmatrix}.$$

Find the average population during the first 350 weeks for the combined population of feral cats and field mice, where the population of feral cats is modelled by the transformation of $y = \sin(t)$ under Q. Give your answer correct to the nearest whole number. (4 marks)

d Over a longer period of time, it is found that the increase and decrease in the population of field mice gets smaller and smaller. The population of field mice over a longer period of time can be modelled by the rule

$$s(t) = 1300e^{-0.004t}\sin\left(\frac{\pi t}{70}\right) + 2200, \quad \text{for all } t \geq 0.$$

i Find the average rate of change between the first two times when the population of field mice is at a maximum. Give your answer correct to one decimal place.(2 marks) ii Find the time, where $t > 50$, in weeks, when the rate of change of the field mouse population is at its greatest positive value. Give your answer correct to the nearest whole number. (2 marks)

iii Over time, the field mouse population approaches a particular value. State this value. (1 mark)

Solutions

1. (1) **B.** Under $y = -2f(x+2)+5$ the x-coordinate is translated by -2 and the y-coordinate is multiplied by -2 before 5 is added, so the image is $(-4,-5)$.

   (2) **D.** Taking logarithms gives $y = \frac{1}{4}[\log_e(2+x) - \log_e(2-x)]$. Hence at $x = 0$, $\frac{dy}{dx} = \frac{1}{4}(\frac{1}{2}+\frac{1}{2}) = \frac{1}{4}$.

   (3) **D.** Substituting $u = 3(x+2)$ gives $du = 3\,dx$ and turns the limits into 3 and 9, so the integral is $-3 \div 3 = -1$.

   (4) **C.** Condition on the colour transferred. The probability is $\dfrac{2}{6}\dfrac{5}{7} + \dfrac{4}{6}\dfrac{4}{7} = \frac{13}{21}$.

   (5) **C.** The derivative is $g'(x) = \log_e(-4x)+1$. Thus $g'(x) = 0$ gives $-4x = e^{-1}$ and hence $x = -\frac{1}{4e}$.

   (6) **A.** For the vertical asymptote, solve $3x-9 = -5$, giving $x = \frac{4}{3}$. Values approaching the original horizontal asymptote are mapped to -4, so the other asymptote is $y = -4$.

   (7) **D.** The integral from 0 to 4 is the one from 0 to 2 plus the one from 2 to 4, so the answer is $-10-(-9) = -1$.

   (8) **E.** Independence gives $\Pr(B \mid A) = \Pr(B \mid A') = \Pr(B)$, so $n = \frac{1}{5}$.

   (9) **C.** Since $f(2) = 29$ and $f'(2) = 10$, the inverse-function rule gives $g'(29) = \frac{1}{10}$.

   (10) **E.** The denominator is zero at $x = -1$, so f is not defined there and cannot be continuous there.

   (11) **B.** $f'(x) = -\frac{15\pi}{100}\sin\left(\frac{\pi x}{100}\right)$. At $x = 50$ the sine factor is 1, so the gradient is $-\frac{3\pi}{20}$.

   (12) **D.** Over an interval symmetric about the origin the odd terms $-3x^3$ and $6x$ contribute nothing, leaving 6 times the width 6.

   (13) **D.** Since $a < 17$, the numerator event is contained in the conditioning event, so $\Pr(X \geq 17 \mid X \geq a) = \dfrac{\Pr(X \geq 17)}{\Pr(X \geq a)}$. Testing the integer thresholds gives $a = 14$.

   (14) **C.** Reflecting in the x-axis negates the rule, and the translation is applied to the value afterwards, giving $y = -f(x)-2$.

   (15) **A.** Differentiating $f(y) = \sin(3x)$ gives $f'(y)\frac{dy}{dx} = 3\cos(3x)$. At the stated point, $y = 3$ and the cosine value is $\frac{\sqrt{2}}{2}$, so $\frac{dy}{dx} = \frac{3\sqrt{2}}{10}$.

   (16) **C.** An antiderivative of f' is f itself, so the integral is $f(1) - f(-2) = -6$.

(17) **D.** Every other rule is defined and continuous across the whole interval. The first is undefined at $x = 2$, which lies inside it.

(18) **E.** $a = \frac{1}{22}$ because the probabilities sum to one. Therefore $\mathrm{E}(X) = a\sum xc_x = \frac{24}{11}$, where c_x is the coefficient of a at x.

(19) **C.** $f'(x) = 2x + 10$ is zero at $x = -5$. The quadratic is strictly decreasing up to this turning point, so the largest valid endpoint is $a = -5$.

(20) **A.** $f(g(x)) = 4x - 14$, and making x the subject gives $\frac{x}{4} + \frac{7}{2}$. Undoing a composite means undoing f first and g second.

2. (a) Using the required working, the answer is $c(0) = 0$ milligrams per litre.

(b) Using the required working, the answer is $c'(t) = 5e^{-0.3t}(1 - 0.3t),\ c'(t) = 0$ when $t = \frac{1}{0.3},\ t \approx 3.33$ hours, $c(3.33) \approx 6.13$ milligrams per litre.

(c) Using the required working, the answer is average$= \frac{1}{6}\int_0^6 5te^{-0.3t}\,dt,\ \approx 4.97$ milligrams per litre.

(d) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $c'(4) \approx -0.30$ milligrams per litre per hour.

(e) Using the required working, the answer is $5te^{-0.3t} = 3,\ t \approx 9.04$ hours.

(f) Using the required working, the answer is $e^{-0.3t}$ decays faster than t grows, $c(t) \to 0$ milligrams per litre.

3. (a) Set the model equal to the required value and solve for the variable. This gives $d(x) = 0$ when $x = \pm 2\sqrt{2}$, width$= 4\sqrt{2} \approx 5.66$ m.

(b) Using the required working, the answer is $d(0) = 2$ m.

(c) Using the required working, the answer is area$= \int_{-2.83}^{2.83}\left(2 - \frac{x^2}{4}\right)dx,\ \approx 7.54$ m^2.

(d) Using the required working, the answer is average depth$= \frac{\text{area}}{\text{width}},\ \approx 1.33$ m.

(e) Using the required working, the answer is volume$=$area$\times 40 \approx 301.70$ m^3.

4. (a) Using the required working, the answer is $V(x) = x(40 - 2x)(16 - 2x),\ V(x) = 4x^3 - 112x^2 + 640x$.

(b) Using the required working, the answer is Both side lengths must be positive, and the shorter side runs out first, $0 < x < 8$.

(c) Using the required working, the answer is $V'(x) = 12x^2 - 224x + 640$.

(d) Using the required working, the answer is $V'(x) = 0$ when $12x^2 - 224x + 640 = 0,\ x = 3.52,\ 15.15$, Only $x = 3.52$ lies in $0 < x < 8$, and $V''(3.52) < 0$, so it gives the greatest volume, $x \approx 3.52$ cm.

(e) Using the required working, the answer is $V(3.52) \approx 1039.53$ cm^3.

(f) $4x^3 - 112x^2 + 640x = 750,\ x \approx 1.59$ and $x \approx 5.70$, The volume is at least 750 for $1.59 \le x \le 5.70$

5. (a) Using the required working, the answer is

$$\int_0^5 kx^2(5 - x)\,dx = 1,\ k\left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_0^5 = k \times \frac{625}{12} = 1,\ k = \frac{12}{625}$$

.

(b) Using the required working, the answer is $\Pr(X < 2) = \int_0^2 \frac{12}{625}x^2(5 - x)\,dx,\ \approx 0.1792$.

(c) Using the required working, the answer is

$$E(X) = \int_0^5 x \times \frac{12}{625}x^2(5 - x)\,dx,\ E(X) = 3 \text{ minutes}$$

.

(d) Using the required working, the answer is $E\left(X^2\right)=\int_0^5 x^2\times\frac{12}{625}x^2(5-x)\,dx=10$,
$\text{sd}(X)=\sqrt{E\left(X^2\right)-[E(X)]^2}=1,\ \approx 1.00$ minutes.

(e) Using the required working, the answer is $\int_0^m \frac{12}{625}x^2(5-x)\,dx=0.5,\ m\approx 3.07$ minutes.

6. (a) (i) Using the required working, the answer is $r(0)=2200$.

(ii) Using the required working, the answer is minimum$=900$, maximum$=3500$.

(iii) Using the required working, the answer is period$=\frac{2\pi}{\frac{\pi}{70}}=140$ weeks.

(b) (i) $a=\frac{2300-500}{2}=900$, The maximum is half a period after the minimum, so the period is $2(80-10)=140,\ b=\frac{2\pi}{140}=\frac{\pi}{70}$

(ii) Using the required working, the answer is Both populations have period 140, so the sum is a single sinusoid, amplitude$=1218.49$, maximum$=3600+1218.49\approx 4818$.

(iii) The sum has the same period as each population, 140 weeks

(c) Using the required working, the answer is Under Q the image of $y=\sin(t)$ is $y=900\sin\left(\frac{\pi(t-45)}{90}\right)+1400$, average

$$=\frac{1}{350}\int_0^{350}\left(1300\sin\left(\frac{\pi t}{70}\right)+2200+900\sin\left(\frac{\pi(t-45)}{90}\right)+1400\right)dt,\ \approx 3791$$

.

(d) (i) Use the average-rate formula $\frac{f(b)-f(a)}{b-a}$ over the given interval. This gives $s'(t)=0$ where

$$\tan\left(\frac{\pi}{70}t\right)=\frac{\frac{\pi}{70}}{0.004},\ t_1\approx 33.02,\quad t_2\approx 173.02$$

, average rate

$$=\frac{s(t_2)-s(t_1)}{t_2-t_1}\approx -3.5 \text{ per week}$$

.

(ii) $s''(t)=0$ where

$$\tan\left(\frac{\pi}{70}t\right)=\frac{2\times 0.004\times\frac{\pi}{70}}{(0.004)^2-\left(\frac{\pi}{70}\right)^2}$$

, The decay makes each later positive peak smaller, so the first one after $t=50$ is the greatest, $t\approx 136$ weeks

(iii) Using the required working, the answer is $e^{-0.004t}\to 0$ as $t\to\infty,\ s(t)\to 2200$.