MATHEMATICS ADVANCED

Written examination: School-based Assessment

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
7	**7**	**71**

- Calculators approved by NESA may be used.
- A reference sheet is provided at the back of this paper.
- Write your Centre Number and Student Number at the top of this page.

Question 1

(7 marks)

a Solve the following inequality

$$(q+1)(q+2)(q+5) \geq 0$$

A
$$-5 \geq q \geq -2, q \geq -1$$

B
$$-8 \leq q \leq 4, q \leq -9$$

C
$$q > -8$$

D
$$-9 > q > 4, q > 8$$

(1 mark)

b Consider the system of equations representing three planes in 3D. How many solutions does it have?

$$3x - 3y - 5z = -35$$
$$x - y - z = -9$$
$$6x - 6y - 12z = -78$$

A Infinitely many solutions (a line of intersection).
B One solution (a single point).
C Infinitely many solutions (a plane).
D No solution (inconsistent system).

(1 mark)

c Sketch the graph of the function

$$f(x) = x^2 - x - 12$$

(1 mark)

d What constant term should you add to the following expression to complete a square?

$$9z^2 + 48z$$

A
$$448$$

B
$$\frac{64}{7}$$

C
$$-5z - 3$$

D
$$64$$

(1 mark)

e Evaluate

$$\log_3\left(\frac{1}{27}\right)$$

A -3
B 5
C -6
D 9

(1 mark)

f Differentiate the following expression with respect to x

$$\frac{-4x-5}{\sqrt{-2x^2-3x-3}}$$

A

$$\frac{16x^2+24x-(4x+3)(4x+5)+24}{2\left(-2x^2-3x-3\right)^{\frac{3}{2}}}$$

B

$$4x^3-2x^2+5x+1$$

C

$$\frac{5\left(16x^2+24x-(4x+3)(4x+5)+24\right)}{2\left(-2x^2-3x-3\right)^{\frac{3}{2}}}$$

D

$$\frac{16x^2+24x-(4x+3)(4x+5)+24}{12\left(-2x^2-3x-3\right)^{\frac{3}{2}}}$$

(1 mark)

g Let

$$f(x)=ax^2+bx^4+4x^3-12$$

If $(0,-12)$ and $(6,-4332)$ are stationary points, find the values of a and b.

A

$$a=-249, b=0$$

B

$$a=-250, b=4$$

C

$$a=-252, b=3$$

D

$$a=-255, b=6$$

(1 mark)

Question 2

a Ecologists are recording the number of Eastern Quolls living in a fenced sanctuary. Let $P(t)$ be the population, in individuals, t years after the release program began. For the initial stage of the study, the model used is

$$P(t)=110e^{0.24t}.$$

i Differentiate the model and show that the instantaneous growth rate is $\frac{dP}{dt} = 36e^{0.30t}$ individuals per year. (1 mark)

ii Evaluate the growth rate at $t = 2$ years and interpret that value in context. (2 marks)

iii Find the first time at which the growth rate is greater than 150 individuals per year. Give your answer correct to one decimal place. (2 marks)

iv Compute the average growth rate over the interval $0 \leq t \leq 5$, and compare it with the instantaneous growth rate at $t = 5$. (2 marks)

v Use differentiation to explain why the growth rate is increasing throughout this stage. (1 mark)

b After 6 years, a feral-cat breach causes an immediate loss of 25% of the Eastern Quolls and the subsequent exponential growth is slower. The revised population model is

$$P(t) = \begin{cases} 110e^{0.24t}, & 0 \leq t \leq 6, \\ 348.21e^{0.16(t-6)}, & t > 6, \end{cases}$$

where t is still measured in years from the initial release.

i Calculate the population immediately after the incursion at $t = 6$. (1 mark) ii Differentiate both branches to obtain a piecewise expression for $\frac{dP}{dt}$. (2 marks)

iii Find the population size and growth rate just before and just after the breach at $t = 6$. State the percentage change in each quantity. (3 marks)

iv Measure time from the breach. After how many years will the population recover to the level it had immediately before $t = 6$? Give your answer correct to one decimal place. (2 marks)

v Suppose the original exponential model had continued with no breach until $t = 10$. Compare that hypothetical population with the actual population at $t = 10$, stating which is larger and by how many individuals. (2 marks)

c At $t = 9$ years, three years after the feral-cat breach, a restoration program adds 40 Eastern Quolls and improves habitat quality. The population is then described by

$$P(t) = \begin{cases} 110e^{0.24t}, & 0 \le t \le 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \le 9, \\ (90e^{2.34} + 40)e^{0.28(t-9)}, & t > 9, \end{cases}$$

with both changes occurring instantaneously at their stated times.

i Calculate the population immediately after the restoration at $t = 9$. (1 mark)

ii Differentiate the three branches and write down the corresponding piecewise function for $\frac{dP}{dt}$. (2 marks) iii Evaluate the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

iv How many years after the restoration program starts will the population be twice its size at the point just after $t = 9$? Give your answer correct to one decimal place. (2 marks)

v If the middle branch had simply continued for all $t > 6$, compare that model with the actual restoration model at $t = 12$. State the population difference and which model gives the larger population. (2 marks)

d In a different response plan, the managers begin a supplementary-feeding program at $t = 9$ years and simultaneously release 60 Eastern Quolls. The resulting model becomes

$$P(t) = \begin{cases} 110e^{0.24t}, & 0 \le t \le 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \le 9, \\ (90e^{2.34} + 60)e^{0.24(t-9)}, & t > 9, \end{cases}$$

where t is measured in years from the beginning of the study.

i Calculate the population immediately after the supplementary-feeding intervention at $t = 9$. (1 mark)

ii Differentiate each branch to form the corresponding piecewise growth-rate function $\dfrac{dP}{dt}$. (2 marks) iii Determine the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

iv Taking $t = 9$ as the start of the feeding program, after how many years will the population double from its value at the point just after $t = 9$? Give your answer correct to one decimal place. (2 marks)

v Compare the population at $t = 12$ for this feeding model with the population obtained by extending the middle branch with no support. State which population is larger and by how many individuals. (2 marks)

Question 3

Let $f(x) = 2x^2 + 12x + 19$ and $g(x) = x^2 + 7x + 13$.

a State the domain of f. (1 mark)

b Differentiate $f(x)$ to find $f'(x)$. (1 mark) c State the range of f. (1 mark)

d Find the equation of the tangent line to $y = f(x)$ at $x = -4$. (2 marks)

Question 4

A medication pump delivers a drug at a rate, in milligrams per minute, modelled by

$$\frac{dM}{dt} = 100e^{-\frac{3t}{10}},$$

where t is the time in minutes after the pump starts. At $t = 0$, there are 20 milligrams of the drug present.

a Find the total amount of drug delivered during the first 8 minutes. (2 marks)

b Find an expression for the amount $M(t)$ of drug present after t minutes. (2 marks)

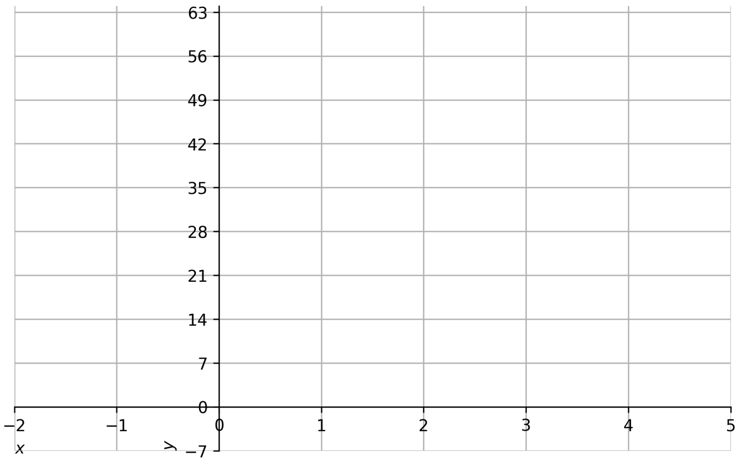
c Determine the amount of drug present after 8 minutes. (1 mark)

d Find the total amount of drug that will be delivered if the pump runs indefinitely. (2 marks)

Question 5

Let $f : R \to R,\ f(x) = x^4 - 6x^3 + 5x^2 + 12x$.

Sketch the graph of $f(x)$ on the following grid.



(4 marks)

Question 6

The quadratic function $f(x) = 2x^2 - 8x + 6$ has x-intercepts P and Q.

a State the coordinates of P and Q. (1 mark)

b State the axis of symmetry and the coordinates of the turning point of the graph.(2 marks)

c Find the exact area enclosed by the graph of $y = f(x)$ and the line segment PQ.(2 marks)

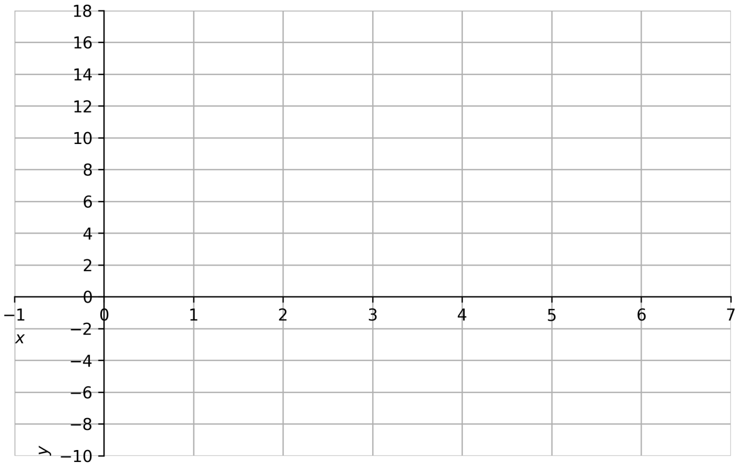
d State the minimum value of f and the value of x for which it occurs. (1 mark)

Question 7

A graph is defined by the rule $y = (x-5)(x-1)$. The graph is transformed in the following order:

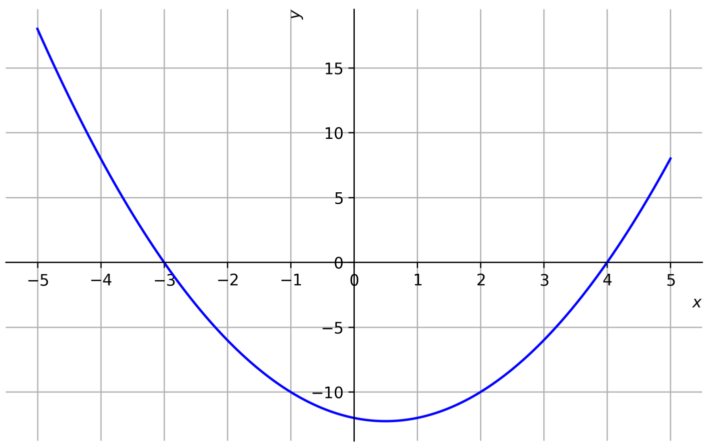
1. horizontal dilation by a factor of 2
2. vertical dilation by a factor of $\frac{1}{-3}$
3. horizontal translation 1 units to the right

The transformed function is labeled Y.
Sketch the graphs of y and Y in the following grid.



(4 marks)

Solutions

1. (a) A
   (b) A
   (c) 

   (d) D
   (e) A
   (f) A
   (g) C

2. (a) (i) Differentiate the given function to obtain $\frac{132e^{\frac{6t}{25}}}{5}$.
   (ii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives At $t = 2,\ \frac{dP}{dt} = 42.66$ individuals per year, so the quoll population is increasing at this instantaneous rate after 2 years..

(iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 7.2.

(iv) Average growth rate= 51.04, instantaneous rate at $t=5$ is 87.65. The instantaneous rate at $t=5$ is larger.

(v) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $\frac{792e^{\frac{6t}{25}}}{125} > 0$, so $\frac{dP}{dt}$ is increasing for all t.

(b) (i) Using the required working, the answer is 348.21.

(ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} \frac{132e^{\frac{6t}{25}}}{5}, & 0 \le t \le 6 \\ \frac{66e^{\frac{36}{25}}e^{\frac{4t}{25}-\frac{24}{25}}}{5}, & t > 6. \end{cases}$$

.

(iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives Population:
$P(6) immediately before = 464.28,\ P(6) immediately after = 348.21$ so there is a 25.0% decrease. Growth rate:
$P'(6) immediately before = 111.43,\ P'(6) immediately after = 55.71$ so there is a 55.0% decrease..

(iv) Set the model equal to the required value and solve for the variable. This gives 1.8.

(v) Set the model equal to the required value and solve for the variable. This gives Without the incursion there would be 552.18 more quolls at $t=10$, so the actual model has fewer quolls..

(c) (i) Using the required working, the answer is 602.73.

(ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} \frac{132e^{\frac{6t}{25}}}{5}, & 0 \le t \le 6 \\ \frac{66e^{\frac{36}{25}}e^{\frac{4t}{25}-\frac{24}{25}}}{5}, & 6 < t \le 9 \\ \frac{6\left(40+\frac{165e^{\frac{48}{25}}}{2}\right)e^{\frac{6t}{25}-\frac{54}{25}}}{25}, & t > 9. \end{cases}$$

.

(iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population:
$P(9) immediately before = 562.73,\ P(9) immediately after = 602.73$, an increase of 7.11%. Growth rate:
$P'(9) immediately before = 90.04,\ P'(9) immediately after = 144.65$, an increase of 60.66%.

(iv) Using the required working, the answer is 2.9.

(v) The actual model has 328.85 more quolls at $t=12$ than the no-intervention model.

(d) (i) Using the required working, the answer is 622.73.

(ii) Differentiate each branch separately on its stated interval. This gives

$$\frac{dP}{dt} = \begin{cases} \frac{132e^{\frac{6t}{25}}}{5}, & 0 \le t \le 6 \\ \frac{66e^{\frac{36}{25}}e^{\frac{4t}{25}-\frac{24}{25}}}{5}, & 6 < t \le 9 \\ \frac{\left(60+\frac{165e^{\frac{48}{25}}}{2}\right)e^{\frac{t}{5}-\frac{9}{5}}}{5}, & t > 9. \end{cases}$$

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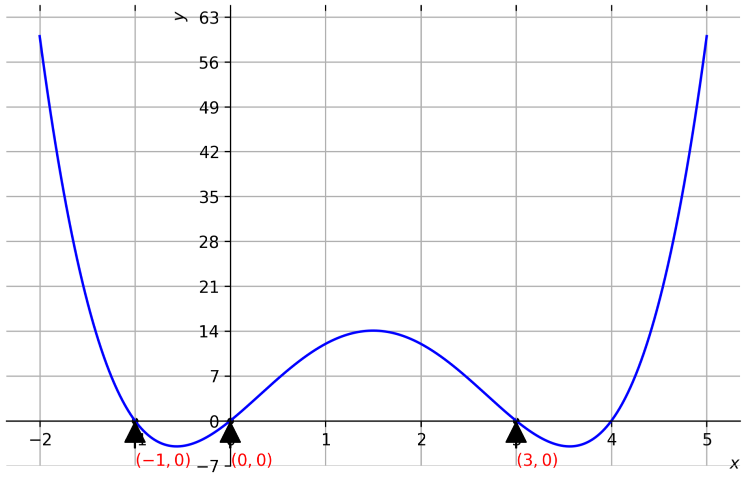
(iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population:
$P(9) immediately before = 562.73,\ P(9) immediately after = 622.73$, an increase of

10.66%. Growth rate: $P'(9) immediately before = 90.04,\ P'(9) immediately after = 124.55$, an increase of 38.33%.

(iv) Using the required working, the answer is 3.5.

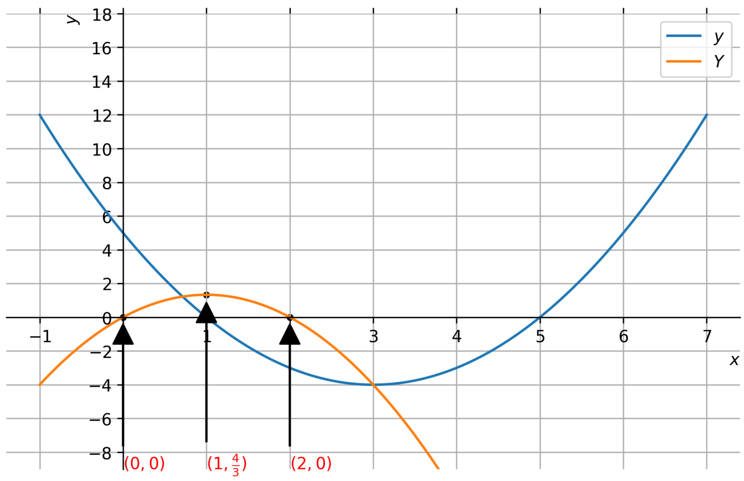
(v) The supplementary-feeding model gives 225.27 more quolls at $t = 12$ years.

3. (a) Identify all allowable input values. The domain is $\mathbb{R}$.
   (b) Differentiate the given function to obtain $4x + 12$.
   (c) Determine the possible output values from the function. The range is $[1, \infty)$.
   (d) Find the gradient from the derivative at the given point, then use point-slope form. This gives $y = -4x - 13$.
4. (a) Using the required working, the answer is $\frac{1000}{3} - \frac{1000}{3e^{\frac{12}{5}}}$.
   (b) Using the required working, the answer is $M(t) = 20 - \frac{1000e^{-\frac{3t}{10}}}{3}$.
   (c) Using the required working, the answer is -10.24.
   (d) Using the required working, the answer is 333.333.



5.

6. (a) $(1, 0)$ and $(3, 0)$
   (b) Axis of symmetry: $x = 2$; turning point: $(2, -2)$
   (c) $\frac{8}{3}$
   (d) Minimum = -2 when $x = 2$



7.