MATHEMATICS ADVANCED

Written examination: School-based Assessment

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
6	**6**	**90**

- Calculators approved by NESA may be used.
- A reference sheet is provided at the back of this paper.
- Write your Centre Number and Student Number at the top of this page.

Question 1

(9 marks)

a If $f(x)$ is defined as

$$f(x) = -2x - 5$$

and $g(x)$ is defined as

$$g(x) = -3x^2 - 4x + 4$$

use the chain rule to find $g^{'}(f(x))$.

A
$$-24x - 52$$

B
$$-3x^2 - 2x + 9$$

C
$$-48x - 104$$

D
$$-3x^2 - 28x - 48$$

(1 mark)

b If $f(x)$ is defined as

$$f(x) = 2x - 3$$

$g(x)$ is defined as

$$g(x) = 2x^2 - x - 3$$

and $h(x)$ is defined as

$$h(x) = -x^2 + 2x + 4$$

use the chain rule to find $h^{'}(g(f(x)))$.

A
$$2x^2 - 3x$$

B
$$-256x^3 + 1248x^2 - 1896x + 884$$

C
$$-1792x^3 + 8736x^2 - 13272x + 6188$$

D
$$-256x^3 + 1250x^2 - 1897x + 881$$

(1 mark)

c If $f(x)$ is defined as

$$f(x) = |4x - 6|$$

evaluate $f(-1)$.

A 6
B 11
C 14
D 10

(1 mark)

d Differentiate the following expression with respect to w:

$$\left(4w^3-5w^2-5w+5\right)^{\frac{7}{2}}$$

A

$$7\left(42w^2-35w-\frac{35}{2}\right)\left(4w^3-5w^2-5w+5\right)^{\frac{5}{2}}$$

B

$$2w^2+2w-4$$

C

$$\frac{\left(42w^2-35w-\frac{35}{2}\right)\left(4w^3-5w^2-5w+5\right)^{\frac{5}{2}}}{5}$$

D

$$\left(42w^2-35w-\frac{35}{2}\right)\left(4w^3-5w^2-5w+5\right)^{\frac{5}{2}}$$

(1 mark)

e Sketch the graph of the function

$$f(x)=\left|(x-4)(x+3)\right|$$

(1 mark)

f Differentiate the following expression with respect to q

$$\frac{-q-3}{3q^2+3q-4}$$

A

$$\frac{3q^2+18q+13}{9q^4+18q^3-15q^2-24q+16}$$

B

$$-3q^3-3q^2-q-4$$

C

$$\frac{5\left(3q^2+18q+13\right)}{9q^4+18q^3-15q^2-24q+16}$$

D

$$\frac{3q^2+18q+13}{11\left(9q^4+18q^3-15q^2-24q+16\right)}$$

(1 mark)

g Let

$$f(x)=x^2+x-5$$

Determine the nature of its stationary point.

A maximum
B minimum
C point of inflection
D not a stationary point

(1 mark)

h Differentiate the following expression with respect to u

$$4e^{-8u^2y}\tan^2\left(u^2y^2\right)$$

A

$$16uy\left(\frac{y}{\cos^2\left(u^2y^2\right)}-4\tan\left(u^2y^2\right)\right)e^{-8u^2y}\tan\left(u^2y^2\right)$$

B

$$\frac{16uy\left(\frac{y}{\cos^2\left(u^2y^2\right)}-4\tan\left(u^2y^2\right)\right)e^{-8u^2y}\tan\left(u^2y^2\right)}{19}$$

C

$$96uy\left(\frac{y}{\cos^2\left(u^2y^2\right)}-4\tan\left(u^2y^2\right)\right)e^{-8u^2y}\tan\left(u^2y^2\right)$$

D

$$-2e^{uy^2}\sin\left(4u^2y^2\right)$$

(1 mark)

i Differentiate the following expression with respect to n

$$4\cos^2\left(n^2u^2\right)-3\cos^2\left(5n^2u^2\right)$$

A

$$60nu^2\left(-2\sin\left(2n^2u^2\right)+\frac{15\sin\left(10n^2u^2\right)}{2}\right)$$

B

$$4nu^2\left(-2\sin\left(2n^2u^2\right)+\frac{15\sin\left(10n^2u^2\right)}{2}\right)$$

C

$$2\cos^2\left(2n^2u^2\right)$$

D

$$\frac{nu^2\left(-2\sin\left(2n^2u^2\right)+\frac{15\sin\left(10n^2u^2\right)}{2}\right)}{4}$$

(1 mark)

Question 2

a A vibration sensor is attached to a reference marker on a laboratory calibration platform. For the first trial, the marker's vertical displacement from its centre line is modelled by

$$y(t)=\frac{3\cos\left(5\sqrt{10}t\right)}{20}.$$

Here t is measured in seconds and y in metres. This trial is called Model A.

i Differentiate $y(t)$ to find $y'(t)$. (1 mark) ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Determine the maximum speed during the first full period, and state the times at which it occurs. (2 marks)

iv Differentiate again to find $y''(t)$, then evaluate $y''(0.10)$. (2 marks)

v State the amplitude and period of Model A. (1 mark)

b The same calibration platform is then run at a faster setting while keeping the same maximum displacement. The displacement is modelled by

$$y(t) = \frac{3\cos(20t)}{20}.$$

This trial is called Model B.

i State the amplitude and period of Model B. (1 mark)

ii Express the period of Model B as a percentage of the period of Model A, correct to one decimal place. (2 marks) iii Differentiate $y(t)$ to find $y'(t)$, then determine the maximum speed during the first full period and the times at which it occurs. (2 marks)

iv Find the greatest magnitude of $y''(t)$ for Model B and compare it with the corresponding value for Model A. (2 marks)

v State the range of Model B. (1 mark)

c A lighter sensor mount is tested next. The recorded displacement has a smaller maximum displacement and a different period:

$$y(t) = \frac{\cos\left(8\sqrt{5}t\right)}{10}.$$

This trial is called Model C.

i Express the period of Model C as a percentage of the period of Model B, correct to one decimal place. (2 marks)

ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Find the maximum speed and the greatest magnitude of $y''(t)$. (2 marks) iv State the amplitude and range of Model C. (1 mark)

d In a final trial, the marker starts at the centre line and initially moves upward. Its displacement is modelled by

$$y(t) = \frac{\sin(20t)}{10}.$$

This trial is called Model D.

i Differentiate to obtain expressions for $y'(t)$ and $y''(t)$. (2 marks)

ii Find the first time after $t = 0$ at which $y(t)$ reaches its maximum value, and state that value. (2 marks)

iii Find the first positive value of t for which $y(t) = 0$ again, and give the corresponding value of $y'(t)$. (2 marks)

iv State the amplitude and period of Model D. (1 mark)

v Determine the maximum magnitude of $y''(t)$. (1 mark)

Question 3

a Wildlife rangers are auditing the Eastern Quoll population in a secure heathland enclosure. Let $P(t)$ denote the number of quolls t years after the monitoring program starts. For the initial phase, the population is modelled by

$$P(t) = 130e^{0.26t}$$

i By differentiating the model, establish that the instantaneous rate of change is $\dfrac{dP}{dt} = \frac{169e^{\frac{13t}{50}}}{5}$ individuals per year. (1 mark)

ii Determine the instantaneous growth rate when $t = 2$, and explain what that value means for the population. (2 marks)

iii Find the earliest time for which the growth rate is above 150 individuals per year. Give your answer correct to one decimal place. (2 marks)

iv Work out the average rate of population growth over the first 5 years, and compare it with the instantaneous rate at $t = 5$. (2 marks)

v Use the derivative information to justify why the growth rate keeps rising throughout this phase. (1 mark)

b At $t = 6$ years, a predator breach causes an immediate 25% fall in the Eastern Quoll population, and the continuing exponential growth becomes slower. The updated model is

$$P(t) = \begin{cases} 130e^{0.26t}, & 0 \le t \le 6, \\ 463.99e^{0.2(t-6)}, & t > 6, \end{cases}$$

with t still measured from the original release date.

i Rewrite the branch for $t > 6$ in simplified form. (1 mark)

ii Differentiate each branch and state the resulting piecewise formula for $\frac{dP}{dt}$.(2 marks)

iii Calculate the population and the growth rate immediately before and immediately after $t = 6$. State the percentage decrease in each. (3 marks)

iv Measured from the breach, how long does it take for the population to recover to its pre-breach level? Give your answer correct to one decimal place. (2 marks)

v Assume the original exponential rule had continued unchanged until $t = 10$. Compare that hypothetical population with the actual population at $t = 10$, stating which one is larger and by how many individuals. (2 marks)

c At $t = 9$ years, three years after the feral-cat breach, a restoration program adds 40 Eastern Quolls and improves habitat quality. The population is then described by

$$P(t) = \begin{cases} 130e^{0.26t}, & 0 \leq t \leq 6, \\ 463.99e^{0.2(t-6)}, & 6 < t \leq 9, \\ 885.44e^{0.22(t-9)}, & t > 9, \end{cases}$$

with both changes occurring instantaneously at their stated times.

i Calculate the population immediately after the restoration at $t = 9$. (1 mark) ii Differentiate the three branches and write down the corresponding piecewise function for $\frac{dP}{dt}$. (2 marks)

iii Evaluate the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

iv How many years after the restoration program starts will the population be twice its size at the point just after $t = 9$? Give your answer correct to one decimal place. (2 marks)

v If the middle branch had simply continued for all $t > 6$, compare that model with the actual restoration model at $t = 12$. State the population difference and which model gives the larger population. (2 marks)

d In a different response plan, the managers begin a supplementary-feeding program at $t = 9$ years and simultaneously release 60 Eastern Quolls. The resulting model becomes

$$P(t) = \begin{cases} 130e^{0.26t}, & 0 \leq t \leq 6, \\ 463.99e^{0.2(t-6)}, & 6 < t \leq 9, \\ 905.44e^{0.3(t-9)}, & t > 9, \end{cases}$$

where t is measured in years from the beginning of the study.

i Calculate the population immediately after the supplementary-feeding intervention at $t = 9$. (1 mark) ii Differentiate each branch to form the corresponding piecewise growth-rate function $\frac{dP}{dt}$. (2 marks)

iii Determine the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

iv Taking $t = 9$ as the start of the feeding program, after how many years will the population double from its value at the point just after $t = 9$? Give your answer correct to one decimal place. (2 marks)

v Compare the population at $t = 12$ for this feeding model with the population obtained by extending the middle branch with no support. State which population is larger and by how many individuals. (2 marks)

Question 4

The volume of water in a reservoir (in kilolitres) after t hours is given by the function $V(t) = -t^2 + 10t$.

a Calculate the rate at which the volume is changing at t = 2 hours. (1 mark)

b Find the time at which the volume reaches its maximum. (1 mark) c Find the maximum volume of the reservoir. (1 mark)

Question 5

The depth of water at a harbour is modelled by the trigonometric function

$$d(t) = 5 + \frac{3}{2}\sin\left(\frac{\pi t}{6}\right),$$

where $d(t)$ is the depth in metres and t is the number of hours after midnight.

a Differentiate $d(t)$ to find the rate at which the depth is changing. (1 mark)

b Find the rate of change of the water depth at $t = 2$ hours and state whether the tide is rising or falling. (2 marks)

c Determine the first time after midnight that the water reaches high tide. (2 marks)

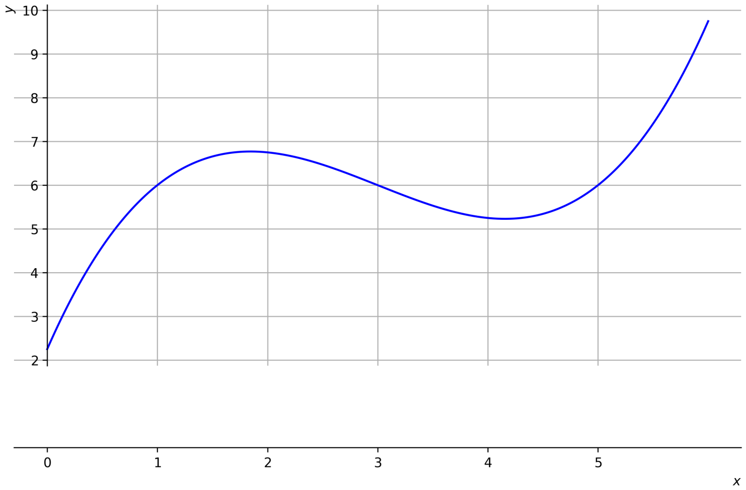
d During the first 12 hours, when is the water level falling most rapidly, and what is that rate? (2 marks)

Question 6

Diala is designing a fence for her garden. The outline of a part of the fence should look like the curve of the following function:

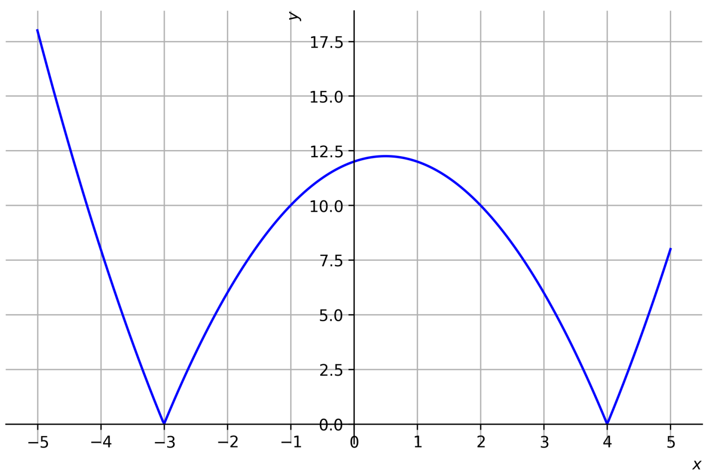
$$f(x) = \frac{1}{4}(x-1)(x-3)(x-5)+6$$

where x and $f(x)$ are in metres.



Find the rate of change of $f(x)$ at $x = 5$ m. Write your answer correct to 3 significant figures. (2 marks)

Solutions

1. (a) A
   (b) B
   (c) D
   (d) D
   (e) 

(f) A

(g) B

(h) A

(i) B

2. (a) (i) Differentiate the given function to obtain $y'(t) = -\frac{3\sqrt{10}\sin\left(5\sqrt{10}t\right)}{4}$ m s^{-1}.

(ii) Using the required working, the answer is $t = \frac{\sqrt{10}\pi}{100} \approx 0.0993$ s.

(iii) Using the required working, the answer is maximum speed $= \frac{3\sqrt{10}}{4} \approx 2.37$ m s^{-1}, at $t = \frac{\sqrt{10}\pi}{100}$ and $t = \frac{3\sqrt{10}\pi}{100}$.

(iv) Differentiate the given function to obtain $y''(t) = -\frac{75\cos\left(5\sqrt{10}t\right)}{2}$, $y''(0.10) = 0.39$.

(v) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\sqrt{10}\pi}{25} \approx 0.3974 \text{ s}$$

.

(b) (i) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(ii) Using the required working, the answer is $T_B = \frac{\pi}{10} \approx 0.314$ s, which is 79.1% of the period of Model A..

(iii) Differentiate the given function to obtain $y'(t) = -3\sin(20t)$, maximum speed $= 3.00$ m s^{-1} at $t = \frac{\pi}{40}$ and $t = \frac{3\pi}{40}$.

(iv) Using the required working, the answer is greatest magnitude of $y''(t)$ for Model B $= 60 = 60$ m s^{-2}, while for Model A it is $\frac{75}{2} = 37.5$ m s^{-2}. So Model B has the larger maximum second derivative..

(v) Determine the possible output values from the function. The range is $-0.15 \leq y(t) \leq 0.15$.

(c) (i) Using the required working, the answer is $T = \frac{\sqrt{5}\pi}{20} \approx 0.3512$ s, which is 111.8% of the period of Model B..

(ii) Using the required working, the answer is $t = \frac{\sqrt{5}\pi}{80} \approx 0.0878$ s.

(iii) Using the required working, the answer is maximum speed $= \frac{4\sqrt{5}}{5} \approx 1.79$ m s^{-1}, greatest magnitude of $y''(t) = 32 \approx 32$ m s^{-2}.

(iv) Using the required working, the answer is Amplitude

$$= 0.10 \text{ m}, \qquad -0.10 \leq y(t) \leq 0.10$$

.

(d) (i) Differentiate the given function to obtain

$$y'(t) = 2\cos(20t), \qquad y''(t) = -40\sin(20t)$$

.

(ii) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives

$$t = \frac{\pi}{40} \approx 0.0785 \text{ s}, \qquad y_{\max} = \frac{1}{10} \text{ m}$$

.

(iii) Using the required working, the answer is

$$t = \frac{\pi}{20} \approx 0.1571 \text{ s}, \qquad y'(t) = -2 \text{ m s}^{-1}$$

.

(iv) Using the required working, the answer is Amplitude

$$= 0.1 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(v) Using the required working, the answer is maximum magnitude of $y''(t) = 40.0 = 40$ m s^{-2}.

3. (a) (i) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $\frac{169e^{\frac{13t}{50}}}{5}$.

(ii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives At $t = 2$, $\frac{dP}{dt} = 56.85$ individuals per year, so the quoll population is increasing at this instantaneous rate after 2 years..

(iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 5.7.

(iv) Average growth rate$= 69.4$, instantaneous rate at $t = 5$ is 124.02. The instantaneous rate at $t = 5$ is larger.

(v) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $\frac{2197e^{\frac{13t}{50}}}{250} > 0$, so $\frac{dP}{dt}$ is increasing for all t.

(b) (i) Using the required working, the answer is $P(t) = 463.99e^{0.2(t-6)}$.

(ii) Differentiate each branch separately on its stated interval. This gives

$$\frac{dP}{dt} = \begin{cases} \frac{169e^{\frac{13t}{50}}}{5}, & 0 \le t \le 6 \\ \frac{39e^{\frac{39}{25}}e^{\frac{t}{5}-\frac{6}{5}}}{2}, & t > 6. \end{cases}$$

.

(iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(6) immediatelybefore = 618.65,\ P(6) immediatelyafter = 463.99$ so there is a 25.0% decrease. Growth rate: $P'(6) immediatelybefore = 160.85,\ P'(6) immediatelyafter = 92.8$ so there is a 55.0% decrease..

(iv) Set the model equal to the required value and solve for the variable. This gives 1.4.

(v) Using the required working, the answer is Without the incursion there would be 717.67 more quolls at $t = 10$, so the actual model has fewer quolls..

(c) (i) Using the required working, the answer is 885.44.

(ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} \frac{169e^{\frac{13t}{50}}}{5}, & 0 \le t \le 6 \\ \frac{39e^{\frac{39}{25}}e^{\frac{t}{5}-\frac{6}{5}}}{2}, & 6 < t \le 9 \\ \frac{11\left(40+\frac{195e^{\frac{54}{25}}}{2}\right)e^{\frac{11t}{50}-\frac{99}{50}}}{50}, & t > 9. \end{cases}$$

.

(iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(9) immediatelybefore = 845.44,\ P(9) immediatelyafter = 885.44$, an increase of 4.73%. Growth rate: $P'(9) immediatelybefore = 169.09,\ P'(9) immediatelyafter = 194.8$, an increase of 15.2%.

(iv) Using the required working, the answer is 3.2.

(v) The actual model has 172.65 more quolls at $t = 12$ than the no-intervention model.

(d) (i) Using the required working, the answer is 905.44.

(ii) Differentiate each branch separately on its stated interval. This gives

$$\frac{dP}{dt} = \begin{cases} \frac{169e^{\frac{13t}{50}}}{5}, & 0 \le t \le 6 \\ \frac{39e^{\frac{39}{25}}e^{\frac{t}{5}-\frac{6}{5}}}{2}, & 6 < t \le 9 \\ \frac{3\left(60+\frac{195e^{\frac{54}{25}}}{2}\right)e^{\frac{3t}{10}-\frac{27}{10}}}{10}, & t > 9. \end{cases}$$

.

(iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(9) immediately before = 845.44,\ P(9) immediately after = 905.44$, an increase of 7.1%. Growth rate: $P'(9) immediately before = 169.09,\ P'(9) immediately after = 271.63$, an increase of 60.65%.

(iv) Using the required working, the answer is 2.3.

(v) The supplementary-feeding model gives 686.53 more quolls at $t = 12$ years.

4. (a) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 6.

   (b) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 5.

   (c) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 25.

5. (a) Differentiate the given function to obtain $d'(t) = \frac{\pi\cos\left(\frac{\pi t}{6}\right)}{4}$.

   (b) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $d'(2) = \frac{\pi}{8} \approx 0.393$ m h^{-1}, so the tide is rising..

   (c) Set the model equal to the required value and solve for the variable. This gives $t = 3$ hours.

   (d) The water level falls most rapidly at $t = 6$ hours, with $d'(6) = -\frac{\pi}{4} \approx -0.785$ m h^{-1}.

6. Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 2.00.