MATHMATICAL METHODS

Written examination: SAC

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
7	**7**	**68**

- Students are to write in blue or black pen.
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book
- Formula sheet
- Working space is provided throughout the book

Instructions

- Write your **student number** in the space provided above on this page
- Unless otherwise indicated, the diagrams in this book are not drawn to scale
- All written responses must be in English

At the end of the examination

- You may keep the formula sheet

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1

(6 marks)

a Solve the following inequality

$$p^2 - p - 30 \geq 0$$

A

$$p \leq -9, p \geq 5$$

B

$$p \leq -4, p \geq 3$$

C

$$p \leq -5, p \geq 6$$

D

$$p \leq -1, p \geq 10$$

(1 mark)

b Find the value of a that will make the polynomial

$$ax + 10x^3 + 14x^2 - 10$$

divisible by

$$2x + 2$$

A 18
B -9
C 8
D -6

(1 mark)

c If $f(x)$ is defined as

$$f(x) = x + 3$$

$g(x)$ is defined as

$$g(x) = 5x^2 + 5x + 4$$

and $h(x)$ is defined as

$$h(x) = 3x^2 - 4x - 3$$

use the chain rule to find $g'(f(h(x)))$.

A

$$3600x^3 - 7200x^2 + 3800x - 400$$

B

$$180x^3 - 360x^2 + 190x - 20$$

C

$$180x^3 - 355x^2 + 195x - 16$$

D

$$5x^2 + 4x + 1$$

(1 mark)

d Differentiate the following expression with respect to s

$$-\log\left(5s^2 + 5s - 2\right)$$

A

$$\frac{5\left(-2s-1\right)}{13\left(5s^2+5s-2\right)}$$

B

$$\frac{5\left(-2s-1\right)}{5s^2+5s-2}$$

C

$$5s^3-4s^2-4s-4$$

D

$$\frac{60\left(-2s-1\right)}{5s^2+5s-2}$$

(1 mark)

e For the function

$$f(x)=\frac{x^3}{3}-2x^2-x+5$$

find the two points on the graph of $f(x)$ at which the tangents of the graph are parallel to the line connecting the two points $(4,2)$ and $(8,-14)$.

A 1 and 3

B 37 and -20

C -16 and 23

D There are no solutions

(1 mark)

f For the curve defined by

$$2x^3+2x^2+3x-5$$

find the equation of the tangent line at $x=5$.

A

$$y=\frac{173x}{19}+\frac{5025}{19}$$

B

$$y=173x-1354$$

C

$$y=173x-555$$

D

$$y=1384x-6610$$

(1 mark)

Question 2

a A vibration sensor is attached to a reference marker on a laboratory calibration platform. For the first trial, the marker's vertical displacement from its centre line is modelled by

$$y(t)=\frac{3\cos\left(5\sqrt{10}t\right)}{20}.$$

Here t is measured in seconds and y in metres. This trial is called Model A.

i Differentiate $y(t)$ to find $y'(t)$. (1 mark)

ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Determine the maximum speed during the first full period, and state the times at which it occurs. (2 marks)

iv Differentiate again to find $y''(t)$, then evaluate $y''(0.10)$. (2 marks)

v State the amplitude and period of Model A. (1 mark)

b The same calibration platform is then run at a faster setting while keeping the same maximum displacement. The displacement is modelled by

$$y(t) = \frac{3\cos(20t)}{20}.$$

This trial is called Model B.

i State the amplitude and period of Model B. (1 mark)

ii Express the period of Model B as a percentage of the period of Model A, correct to one decimal place. (2 marks) iii Differentiate $y(t)$ to find $y'(t)$, then determine the maximum speed during the first full period and the times at which it occurs. (2 marks)

iv Find the greatest magnitude of $y''(t)$ for Model B and compare it with the corresponding value for Model A. (2 marks)

v State the range of Model B. (1 mark)

c A lighter sensor mount is tested next. The recorded displacement has a smaller maximum displacement and a different period:

$$y(t) = \frac{\cos\left(8\sqrt{5}t\right)}{10}.$$

This trial is called Model C.

i Express the period of Model C as a percentage of the period of Model B, correct to one decimal place. (2 marks)

ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Find the maximum speed and the greatest magnitude of $y''(t)$. (2 marks) iv State the amplitude and range of Model C. (1 mark)

d In a final trial, the marker starts at the centre line and initially moves upward. Its displacement is modelled by

$$y(t) = \frac{\sin(20t)}{10}.$$

This trial is called Model D.

i Differentiate to obtain expressions for $y'(t)$ and $y''(t)$. (2 marks)

ii Find the first time after $t = 0$ at which $y(t)$ reaches its maximum value, and state that value. (2 marks)

iii Find the first positive value of t for which $y(t) = 0$ again, and give the corresponding value of $y'(t)$. (2 marks)

iv State the amplitude and period of Model D. (1 mark)

v Determine the maximum magnitude of $y''(t)$. (1 mark)

Question 3

Let $f(x) = -3x^2 - 12x - 16$ and $g(x) = -4x^2 - 12x - 15$.

a State the domain of f. (1 mark)

b Differentiate $f(x)$ to find $f'(x)$. (1 mark)

c Use the derivative to state where the graph of f is increasing and where it is decreasing.(2 marks)

d Find the point on the curve where the tangent is parallel to the line $y = -6x + 1$. (2 marks)

Question 4

For $-1 \leq a \leq 5$, let R be the region under the curve

$$y = 16 - (x-2)^2$$

and above the x-axis between the lines $x = a - 1$ and $x = a + 1$.

a Find an expression for the area $A(a)$ of the region. (2 marks)

b Differentiate $A(a)$ with respect to a. (1 mark) c Find the value of a for which the area is maximised. (2 marks)

d Hence find the maximum area. (1 mark)

Question 5

A weather balloon rises with vertical velocity, in metres per second, given by

$$v(t) = 16e^{-\frac{t}{4}},$$

where t is the time in seconds after release. At $t = 0$, the balloon is 30 metres above the ground.

a Find the vertical displacement of the balloon during the first 5 seconds. (2 marks)

b Find an expression for the height $h(t)$ of the balloon after t seconds. (2 marks)

c Determine the height of the balloon after 5 seconds. (1 mark) d Find the average vertical velocity over the first 5 seconds. (1 mark)

Question 6

Consider the function $f(x) = \frac{-3x-5}{3x-2}$.

a Express $f(x)$ in the form $A + \frac{B}{2-3x}$, where A and B are constants. (1 mark)

b Find the maximal domain of $f(x)$. (1 mark)

c Find the range of $f(x)$. (1 mark)

d If another function $g(x)$ is defined as

$$g(x) = x + 5$$

solve the equation

$$g(f(x)) = 0$$

(2 marks)

Question 7

The line $y = 9 - 14x$ intersects the graph of $y = f(x)$, where $f(x) = 2x^2 - 8x + 13$, at the points P and Q.

a State the coordinates of P and Q. (2 marks)

b State the axis of symmetry and the coordinates of the turning point of the graph. (2 marks)

c Find the exact area enclosed by the graph of $y = f(x)$ and the line segment PQ. (3 marks)

d State the minimum value of f and the value of x for which it occurs. (1 mark)

Solutions

1. (a) C
   (b) D
   (c) B
   (d) B
   (e) A
   (f) C

2. (a) (i) Differentiate the given function to obtain $y'(t) = -\frac{3\sqrt{10}\sin\left(5\sqrt{10}t\right)}{4}$ m s^{-1}.
   (ii) Using the required working, the answer is $t = \frac{\sqrt{10}\pi}{100} \approx 0.0993$ s.
   (iii) Using the required working, the answer is maximum speed $= \frac{3\sqrt{10}}{4} \approx 2.37$ m s^{-1}, at $t = \frac{\sqrt{10}\pi}{100}$ and $t = \frac{3\sqrt{10}\pi}{100}$.
   (iv) Differentiate the given function to obtain $y''(t) = -\frac{75\cos\left(5\sqrt{10}t\right)}{2}$, $y''(0.10) = 0.39$.

(v) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\sqrt{10}\pi}{25} \approx 0.3974 \text{ s}$$

.

(b) (i) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(ii) Using the required working, the answer is $T_B = \frac{\pi}{10} \approx 0.314$ s, which is 79.1% of the period of Model A..

(iii) Differentiate the given function to obtain $y'(t) = -3\sin(20t)$, maximum speed $= 3.00$ m s^{-1} at $t = \frac{\pi}{40}$ and $t = \frac{3\pi}{40}$.

(iv) Using the required working, the answer is greatest magnitude of $y''(t)$ for Model B $= 60 = 60$ m s^{-2}, while for Model A it is $\frac{75}{2} = 37.5$ m s^{-2}. So Model B has the larger maximum second derivative..

(v) Determine the possible output values from the function. The range is $-0.15 \le y(t) \le 0.15$.

(c) (i) Using the required working, the answer is $T = \frac{\sqrt{5}\pi}{20} \approx 0.3512$ s, which is 111.8% of the period of Model B..

(ii) Using the required working, the answer is $t = \frac{\sqrt{5}\pi}{80} \approx 0.0878$ s.

(iii) Using the required working, the answer is maximum speed $= \frac{4\sqrt{5}}{5} \approx 1.79$ m s^{-1}, greatest magnitude of $y''(t) = 32 \approx 32$ m s^{-2}.

(iv) Using the required working, the answer is Amplitude

$$= 0.10 \text{ m}, \qquad -0.10 \le y(t) \le 0.10$$

.

(d) (i) Differentiate the given function to obtain

$$y'(t) = 2\cos(20t), \qquad y''(t) = -40\sin(20t)$$

.

(ii) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives

$$t = \frac{\pi}{40} \approx 0.0785 \text{ s}, \qquad y_{\max} = \frac{1}{10} \text{ m}$$

.

(iii) Using the required working, the answer is

$$t = \frac{\pi}{20} \approx 0.1571 \text{ s}, \qquad y'(t) = -2 \text{ m s}^{-1}$$

.

(iv) Using the required working, the answer is Amplitude

$$= 0.1 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(v) Using the required working, the answer is maximum magnitude of $y''(t) = 40.0 = 40$ m s^{-2}.

3. (a) Identify all allowable input values. The domain is $\mathbb{R}$.

(b) Differentiate the given function to obtain - 6 x - 12.

(c) Using the required working, the answer is increasing for $x < -2$ and decreasing for $x > -2$.

(d) Match the tangent gradient to the required line gradient and solve for the corresponding point(s). This gives $(-1, -7)$.

4. (a) $A(a) = -2a^2 + 8a + \frac{70}{3}$

(b) $A'(a) = 8 - 4a$

(c) $a = 2$

(d) $A_{max} = \frac{94}{3}$

5. (a) Using the required working, the answer is $64 - \frac{64}{e^{\frac{5}{4}}}$.

(b) Using the required working, the answer is $h(t) = 94 - 64e^{-\frac{t}{4}}$.

(c) Using the required working, the answer is 75.66.

(d) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 9.13.

6. (a)

$$-1 + \frac{7}{2 - 3x}$$

(b)

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$$

(c)

$$(-\infty, -1) \cup (-1, \infty)$$

(d)

$$\begin{bmatrix} \frac{5}{4} \end{bmatrix}$$

7. (a) $(-2, 37)$ and $(-1, 23)$

(b) Axis of symmetry: $x = 2$; turning point: $(2, 5)$

(c) $\frac{1}{3}$

(d) Minimum = 5 when $x = 2$