MATHMATICAL METHODS

Written examination: SAC

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
7	**7**	**66**

- Students are to write in blue or black pen.
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book
- Formula sheet
- Working space is provided throughout the book

Instructions

- Write your **student number** in the space provided above on this page
- Unless otherwise indicated, the diagrams in this book are not drawn to scale
- All written responses must be in English

At the end of the examination

- You may keep the formula sheet

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1

(6 marks)

a Solve the following inequality

$$n^2 + 2n - 63 > 0$$

A

$$n < -9, n > 7$$

B

$$n \leq -9, n \geq 10$$

C

$$n < -6, n > 7$$

D

$$n > 3$$

(1 mark)

b Find the value of a that will make the polynomial

$$ax + 3x^3 - 20x^2 + 20$$

divisible by

$$x - 5$$

A 4
B -2
C 21
D -18

(1 mark)

c If $f(x)$ is defined as

$$f(x) = 1 - 5x$$

$g(x)$ is defined as

$$g(x) = -2x^2 - 2x + 3$$

and $h(x)$ is defined as

$$h(x) = 4x^2 + 2x - 5$$

use the chain rule to find $g'(f(h(x)))$.

A

$$-3200x^3 - 2402x^2 + 3838x + 1063$$

B

$$-2x^2 + 3x + 2$$

C

$$-3200x^3 - 2400x^2 + 3840x + 1060$$

D

$$-19200x^3 - 14400x^2 + 23040x + 6360$$

(1 mark)

d Differentiate the following expression with respect to p

$$4\log\left(-3p^2 - 2p + 1\right)$$

A

$$\frac{8(3p+1)}{3p^2+2p-1}$$

B

$$3p^3+p^2-3p-4$$

C

$$\frac{3p+1}{3p^2+2p-1}$$

D

$$\frac{120(3p+1)}{3p^2+2p-1}$$

(1 mark)

e For the function

$$f(x)=\frac{x^3}{3}-3x^2+10x-4$$

find the two points on the graph of $f(x)$ at which the tangents of the graph are parallel to the line connecting the two points $(-2,1)$ and $(2,5)$.

A There are no solutions

B -3 and 16

C 3 and 3

D 9 and -36

(1 mark)

f For the curve defined by

$$-5x^3+2x^2+4x+5$$

find the equation of the tangent line at $x=5$.

A

$$y=1205-351x$$

B

$$y=5112-351x$$

C

$$y=18755-3861x$$

D

$$y=-\frac{351x}{10}-\frac{749}{2}$$

(1 mark)

Question 2

a A vibration sensor is attached to a reference marker on a laboratory calibration platform. For the first trial, the marker's vertical displacement from its centre line is modelled by

$$y(t)=\frac{3\cos\left(5\sqrt{10}t\right)}{20}.$$

Here t is measured in seconds and y in metres. This trial is called Model A.

i Differentiate $y(t)$ to find $y'(t)$. (1 mark)

ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Determine the maximum speed during the first full period, and state the times at which it occurs. (2 marks)

iv Differentiate again to find $y''(t)$, then evaluate $y''(0.10)$. (2 marks)

v State the amplitude and period of Model A. (1 mark)

b The same calibration platform is then run at a faster setting while keeping the same maximum displacement. The displacement is modelled by

$$y(t) = \frac{3\cos(20t)}{20}.$$

This trial is called Model B.

i State the amplitude and period of Model B. (1 mark)

ii Express the period of Model B as a percentage of the period of Model A, correct to one decimal place. (2 marks) iii Differentiate $y(t)$ to find $y'(t)$, then determine the maximum speed during the first full period and the times at which it occurs. (2 marks)

iv Find the greatest magnitude of $y''(t)$ for Model B and compare it with the corresponding value for Model A. (2 marks)

v State the range of Model B. (1 mark)

c A lighter sensor mount is tested next. The recorded displacement has a smaller maximum displacement and a different period:

$$y(t) = \frac{\cos\left(8\sqrt{5}t\right)}{10}.$$

This trial is called Model C.

i Express the period of Model C as a percentage of the period of Model B, correct to one decimal place. (2 marks)

ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Find the maximum speed and the greatest magnitude of $y''(t)$. (2 marks) iv State the amplitude and range of Model C. (1 mark)

d In a final trial, the marker starts at the centre line and initially moves upward. Its displacement is modelled by

$$y(t) = \frac{\sin(20t)}{10}.$$

This trial is called Model D.

i Differentiate to obtain expressions for $y'(t)$ and $y''(t)$. (2 marks)

ii Find the first time after $t = 0$ at which $y(t)$ reaches its maximum value, and state that value. (2 marks)

iii Find the first positive value of t for which $y(t) = 0$ again, and give the corresponding value of $y'(t)$. (2 marks)

iv State the amplitude and period of Model D. (1 mark)

v Determine the maximum magnitude of $y''(t)$. (1 mark)

Question 3

Let $f(x) = -2x^4 + 16x^3 - 48x^2 + 64x - 27$ and $g(x) = -2x^4 + 16x^3 - 49x^2 + 58x - 32$.

a State the domain of f. (1 mark)

b Differentiate $f(x)$ to find $f'(x)$. (1 mark)

c State the range of f. (1 mark)

d Find the equation of the tangent line to $y = f(x)$ at $x = 3$. (2 marks)

Question 4

For $-4 \leq a \leq 0$, let R be the region between the curves

$$y = 17 - (x+2)^2 \quad \text{and} \quad y = 1$$

from $x = a - 2$ to $x = a + 2$.

a Find an expression for the area $A(a)$ of the region. (2 marks)

b Differentiate $A(a)$ with respect to a. (1 mark) c Find the value of a for which the area is maximised. (2 marks)

d Hence find the maximum area. (1 mark)

Question 5

A charging station adds energy to a battery at a rate, in watt-hours per hour, modelled by

$$\frac{dE}{dt} = 100e^{-\frac{t}{4}},$$

where t is the time in hours after charging begins. Initially, the battery contains 30 watt-hours of energy.

a Find the total amount of energy added during the first 4 hours. (2 marks)

b Find an expression for the energy $E(t)$ in the battery after t hours. (2 marks)

c Determine the energy in the battery after 4 hours. (1 mark) d Find the total amount of energy that would be added if charging continued indefinitely. (2 marks)

Question 6

Consider the function $f(x) = \frac{x+2}{x-2}$.

a Express $f(x)$ in the form $A + \frac{B}{2-x}$, where A and B are constants. (1 mark)

b Find the maximal domain of $f(x)$. (1 mark)

c Find the range of $f(x)$. (1 mark)

d If another function $g(x)$ is defined as

$$g(x) = -x - 5$$

solve the equation

$$g(f(x)) = 0$$

(2 marks)

Question 7

Let P and Q be the points on the graph of $y = f(x)$, where $f(x) = 2x^2 - 4x + 5$, with x-coordinates -2 and 4 respectively.

a State the coordinates of P and Q. (1 mark)

b Find the equation of the line segment PQ. (1 mark)

c State the axis of symmetry and the coordinates of the turning point of the graph. (2 marks)

d Find the exact area enclosed by the graph of $y = f(x)$ and the line segment PQ. (2 marks)

Solutions

1. (a) A
   (b) C
   (c) C
   (d) A
   (e) C
   (f) A

2. (a) (i) Differentiate the given function to obtain $y'(t) = -\frac{3\sqrt{10}\sin\left(5\sqrt{10}t\right)}{4}$ m s^{-1}.
   (ii) Using the required working, the answer is $t = \frac{\sqrt{10}\pi}{100} \approx 0.0993$ s.
   (iii) Using the required working, the answer is maximum speed $= \frac{3\sqrt{10}}{4} \approx 2.37$ m s^{-1}, at $t = \frac{\sqrt{10}\pi}{100}$ and $t = \frac{3\sqrt{10}\pi}{100}$.
   (iv) Differentiate the given function to obtain $y''(t) = -\frac{75\cos\left(5\sqrt{10}t\right)}{2}$, $y''(0.10) = 0.39$.

(v) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\sqrt{10}\pi}{25} \approx 0.3974 \text{ s}$$

.

(b) (i) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(ii) Using the required working, the answer is $T_B = \frac{\pi}{10} \approx 0.314$ s, which is 79.1% of the period of Model A..

(iii) Differentiate the given function to obtain $y'(t) = -3\sin(20t)$, maximum speed $= 3.00 \text{ m s}^{-1}$ at $t = \frac{\pi}{40}$ and $t = \frac{3\pi}{40}$.

(iv) Using the required working, the answer is greatest magnitude of $y''(t)$ for Model B $= 60 = 60 \text{ m s}^{-2}$, while for Model A it is $\frac{75}{2} = 37.5 \text{ m s}^{-2}$. So Model B has the larger maximum second derivative..

(v) Determine the possible output values from the function. The range is $-0.15 \le y(t) \le 0.15$.

(c) (i) Using the required working, the answer is $T = \frac{\sqrt{5}\pi}{20} \approx 0.3512$ s, which is 111.8% of the period of Model B..

(ii) Using the required working, the answer is $t = \frac{\sqrt{5}\pi}{80} \approx 0.0878$ s.

(iii) Using the required working, the answer is maximum speed $= \frac{4\sqrt{5}}{5} \approx 1.79 \text{ m s}^{-1}$, greatest magnitude of $y''(t) = 32 \approx 32 \text{ m s}^{-2}$.

(iv) Using the required working, the answer is Amplitude

$$= 0.10 \text{ m}, \qquad -0.10 \le y(t) \le 0.10$$

.

(d) (i) Differentiate the given function to obtain

$$y'(t) = 2\cos(20t), \qquad y''(t) = -40\sin(20t)$$

.

(ii) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives

$$t = \frac{\pi}{40} \approx 0.0785 \text{ s}, \qquad y_{\max} = \frac{1}{10} \text{ m}$$

.

(iii) Using the required working, the answer is

$$t = \frac{\pi}{20} \approx 0.1571 \text{ s}, \qquad y'(t) = -2 \text{ m s}^{-1}$$

.

(iv) Using the required working, the answer is Amplitude

$$= 0.1 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(v) Using the required working, the answer is maximum magnitude of $y''(t) = 40.0 = 40 \text{ m s}^{-2}$.

3. (a) Identify all allowable input values. The domain is $\mathbb{R}$.

(b) Differentiate the given function to obtain $-8x^3 + 48x^2 - 96x + 64$.

(c) Determine the possible output values from the function. The range is $(-\infty, 5]$.

(d) Find the gradient from the derivative at the given point, then use point-slope form. This gives $y = 27 - 8x$.

4. (a) $A(a) = -4a^2 - 16a + \frac{128}{3}$

   (b) $A'(a) = -8a - 16$

   (c) $a = -2$

   (d) $A_{max} = \frac{176}{3}$

5. (a) Using the required working, the answer is $400 - \frac{400}{e}$.

   (b) Using the required working, the answer is $E(t) = 430 - 400e^{-\frac{t}{4}}$.

   (c) Using the required working, the answer is 282.85.

   (d) Using the required working, the answer is 400.

6. (a)

$$1 - \frac{4}{2-x}$$

   (b)

$$(-\infty, 2) \cup (2, \infty)$$

   (c)

$$(-\infty, 1) \cup (1, \infty)$$

   (d)

$$\left[\frac{4}{3}\right]$$

7. (a) $(-2, 21)$ and $(4, 21)$

   (b) $y = 21$

   (c) Axis of symmetry: $x = 1$; turning point: $(1, 3)$

   (d) 72