Tutorial: Polynomial division

1 Outline

This tutorial article will discuss the following topics:

- Long division of simple polynomials
- Long division of large polynomials
- Finding the roots of a polynomial

2 It's just like integer division, really

Remember how you used to divide a three digit number by a one digit number? Let's refresh our memories and divide 573 by 3.

$$
\begin{array}{rl}
 & 191 \\
3 & \overline{)573} \\
 & \underline{3} \\
 & 27 \\
 & \underline{27} \\
 & 03 \\
 & \underline{3} \\
 & 0
\end{array}
$$

We start with the highest place value in the divident 573, which is 5, and try dividing it by 3. It works, but leaves 2, so 2 goes to the third. OK, I hope you are starting to remember the mechanics of the division. How about $158 \div 3$?

$$
\begin{array}{rl}
 & 52 \\
3 & \overline{)158} \\
 & \underline{15} \\
 & 08 \\
 & \underline{6} \\
 & 2
\end{array}
$$

Note the remainder here, which is 2. So we can express the division operation like that:

$$\frac{158}{3} = 52 + \frac{2}{3}$$

What about a three-digit number divided by a two-digit number? Let's see an example: $158 \div 13$:

$$
\begin{array}{rl}
 & 12 \\
13 & \overline{)158} \\
 & \underline{13} \\
 & 28 \\
 & \underline{26} \\
 & 2
\end{array}
$$

which amounts to

$$\frac{158}{13} = 12 + \frac{2}{13}$$

The procedure to divide a polynomial by another polynomial uses the same procedure! To understand how this works, let's, for a second, imagine that a number like 158 is some polynomial $x^2 + 5x + 8$. So what I did here was: the unit place value in 158, which is 8, will correspond to the constant of a polynomial. The tens place value, 5, corresponds to the coefficient of x. The hundreds is the coefficient of x^2. Likewise, we can see the divisor 13 as a monomial $x+3$. Now, let me show you what happens when we divide this polynomial by the number 13.

$$
\begin{array}{r|r}
 & x+2 \\ \hline
x+3 & x^2+5x+8 \\
 & -x^2-3x \\ \cline{2-2}
 & 2x+8 \\
 & -2x-6 \\ \cline{2-2}
 & 2
\end{array}
$$

The result is $x+2$, which can be expressed as 12, the outcome of the division $158 \div 13$ we did above! And the remainder is also 2, same as the remainder of the division $158 \div 13$. Do you see the similarity yet? We can write down the outcome of the polynomial division like we did for the integer division:

$$\frac{x^2+5x+8}{x+3} = x+2+\frac{2}{x+3}$$

> R Once you are done with the long division of polynomial, before you move on to the next question, it is a good idea to verify your answer by calculating the product of the divisor and the result (adding a remainder if any).

3 Bigger polynomials

The same process applies with larger polynomials. Let's have an example.

Example 1 Divide x^3+3x^2+9x+1 by $x+2$

Solution:

$$
\begin{array}{r|r}
 & x^2+x+7 \\ \hline
x+2 & x^3+3x^2+9x+1 \\
 & -x^3-2x^2 \\ \cline{2-2}
 & x^2+9x \\
 & -x^2-2x \\ \cline{2-2}
 & 7x+1 \\
 & -7x-14 \\ \cline{2-2}
 & -13
\end{array}
$$

So far we have been discussing polynomial division using polynomials where all coefficients and constants are positive. But nothing changes when a coefficient becomes negative, we just use the same procedure. Let's have a look.

Example 2 Divide x^3+3x^2-9x+1 by $x-2$

Solution:

$$
\begin{array}{r|r}
 & x^2+5x+1 \\ \hline
x-2 & x^3+3x^2-9x+1 \\
 & -x^3+2x^2 \\ \cline{2-2}
 & 5x^2-9x \\
 & -5x^2+10x \\ \cline{2-2}
 & x+1 \\
 & -x+2 \\ \cline{2-2}
 & 3
\end{array}
$$

So the fact that $x-2$ has a negative constant, or x^3+3x^2-9x+1 has a negative coefficient of the x term, did not change how we performed the division. We can write the result of the division as follows, like before:

$$\frac{x^3+3x^2-9x+1}{x-2} = x^2+5x+1+\frac{3}{x-2}$$

Let's have a few more examples.

Example 3 Divide x^2-1 by $x+2$

Solution:

$$\begin{array}{rl}
 & x-2 \\
x+2\big) & \overline{\;x^2 \qquad -1} \\
 & \underline{-x^2-2x} \\
 & \qquad -2x-1 \\
 & \qquad \underline{2x+4} \\
 & \qquad\qquad 3
\end{array}$$

Example 4 Divide x^3+4x^2-10 by $3x-2$

Solution:

$$\begin{array}{rl}
 & \qquad \frac{1}{3}x^2+\frac{14}{9}x+\frac{28}{27} \\
3x-2\big) & \overline{\;x^3 \;+4x^2 \qquad\qquad -10} \\
 & \underline{-x^3+\frac{2}{3}x^2} \\
 & \qquad \frac{14}{3}x^2 \\
 & \qquad \underline{-\frac{14}{3}x^2+\frac{28}{9}x} \\
 & \qquad\qquad \frac{28}{9}x-10 \\
 & \qquad\qquad \underline{-\frac{28}{9}x+\frac{56}{27}} \\
 & \qquad\qquad\qquad -\frac{214}{27}
\end{array}$$

Example 5 Divide $2x^3+1$ by $x+1$

Solution:

$$\begin{array}{rl}
 & \qquad 2x^2-2x+2 \\
x+1\big) & \overline{\;2x^3 \qquad\qquad +1} \\
 & \underline{-2x^3-2x^2} \\
 & \qquad -2x^2 \\
 & \qquad \underline{2x^2+2x} \\
 & \qquad\qquad 2x+1 \\
 & \qquad\qquad \underline{-2x-2} \\
 & \qquad\qquad\qquad -1
\end{array}$$

You can even apply the same process to divide a monomial by another monomial.

Example 6 Divide $5x-1$ by $2x+3$

Solution:

$$\begin{array}{rl}
 & \qquad\quad \frac{5}{2} \\
2x+3\big) & \overline{\;5x \;\; -1} \\
 & \underline{-5x-\frac{15}{2}} \\
 & \qquad -\frac{17}{2}
\end{array}$$

which means

$$\frac{5x-1}{2x+3} = \frac{5}{2} - \frac{\frac{17}{2}}{2x+3}$$

Example 7 Express

$$\frac{6x+5}{2x+3}$$

in the form

$$A + \frac{B}{2x+3}$$

Solution: For that, you just need to apply long division, just like the previous example.

$$
\begin{array}{rrr}
 & & 3 \\ \cline{2-3}
2x+3) & 6x & +5 \\
 & -6x & -9 \\ \cline{2-3}
 & & -4
\end{array}
$$

So $A = 3$ and $B = -4$.
We can also divide a polynomial by a quadratic polynomial using the same procedure.

■ **Example 8** Divide $2x^3 - 10x^2 + x + 1$ by $x^2 - 2x + 1$

Solution:

$$
\begin{array}{rrrrr}
 & & & 2x & -6 \\ \cline{2-5}
x^2-2x+1) & 2x^3 & -10x^2 & +x & +1 \\
 & -2x^3 & +4x^2 & -2x & \\ \cline{2-4}
 & & -6x^2 & -x & +1 \\
 & & 6x^2 & -12x & +6 \\ \cline{3-5}
 & & & -13x & +7
\end{array}
$$

4 Polynomial roots

Polynomial division can help us determine the roots of a polynomial, if we know at least one of the roots. For example, let's say you know one of the two roots of the equation

$$x^2 - 6x + 5 = 0$$

which is $x = 1$. How can you determine the second root? Given we know one of the two factors of the quadratic formula, which is $x - 1$, you can just divide $x^2 - 6x + 5$ by $x - 1$ to find out:

$$
\begin{array}{rrrr}
 & & x & -5 \\ \cline{2-4}
x-1) & x^2 & -6x & +5 \\
 & -x^2 & +x & \\ \cline{2-3}
 & & -5x & +5 \\
 & & 5x & -5 \\ \cline{3-4}
 & & & 0
\end{array}
$$

so the other factor is $x - 5$, so the second root is $x = 5$.

■ **Example 9** If we know that 2 is one of the three roots of the equation $x^3 - 7x + 6 = 0$, find the two other roots.

Solution:
We start by dividing the cubic formula by $x - 2$:

$$
\begin{array}{rrrrr}
 & & x^2 & +2x & -3 \\ \cline{2-5}
x-2) & x^3 & & -7x & +6 \\
 & -x^3 & +2x^2 & & \\ \cline{2-3}
 & & 2x^2 & -7x & \\
 & & -2x^2 & +4x & \\ \cline{3-4}
 & & & -3x & +6 \\
 & & & 3x & -6 \\ \cline{4-5}
 & & & & 0
\end{array}
$$

So now we know that $x^3 - 7x + 6 = (x-2)(x^2 + 2x - 3)$, and what needs to be done is just to solve the quadratic equation $x^2 + 2x - 3 = 0$. It's solutions are 1 and -3.

5 Exercise

Now let's do some exercise problems on www.mathpractice.xyz.
Exercise 1 Click here: Long division of polynomials.