Tutorial: Logarithms

1 Outline

This tutorial article will discuss the following topics:

- Laws of Logarithms
- Evaluation of logarithms
- Simplification of logarithmic expressions

2 Introduction

Remember how exponentials work? Let's have a quick review of exponentials. Why? Well, a logarithm is the *inverse* of an exponential. So let's see how that goes.

An exponential looks like this:

$$a^x = b$$

Here, a was raised to the power of x, and the value is b. So the things we are given here are: the base, a, and the power or index, x.

Now, let's look at this the other way round: let's say we are given b and a, rather than x and a. We want to find the value of x. To motivate your thinking, let's consider a simple example.

■ **Example 1** Find the value of x that satisfies the following equation:

$$2^x = 8$$

The solution for this example is quite obvious: $x = 3$. But what if we get a more tricky situation, like

$$10^x = 8$$

how can we find x here? For that, let's learn about logarithms.

Definition 2.1 For the following exponential with base a,

$$a^x = b$$

the **logarithm function** with base a is defined as

$$\log_a b = x$$

So, to find the value of x in $10^x = 8$, we need to apply the logarithm function to b with base a. If you check your scientific calculator, or the calculator displayed in any of the quizzes of www.mathpractice.xyz, you will find a two "log" functions: $\log_{10}$, which is base 10, and $\log_e$, which is base e, also known as the natural logarithm. We will focus now on the first one, the logarithm with base 10. The natural logarithm will be discussed in a separate tutorial.

Can we apply $\log_{10}$ to find x in $10^x = 8$? Yes we can, because the base of 10 is common in the logarithm and the exponential. So using the calculator, you just need to enter `log(8)` to get your x value. The value you will get is a decmial number 2.302585092994046. Let's now check if this value is indeed x in $10^x = 8$. How to do that? Just type the following in the calculator: `10^2.302585092994046`. The answer you will get is 10.

3 Laws of Logarithms

Just as exponentials have got laws, logarithms have got laws too. Remember, one is the opposite of the other.

Law 3.1 — Exponent of a logarithm.

$$a^{\log_a x} = x$$

This is coming from the definition of the logarithm as follows: if we say

$$\log_a x = b$$

then we can write

$$a^{\log_a x} = a^b$$

and we already have said that $a^b = x$, which is just a restatement of $\log_a x = b$.

Law 3.2 — Logarithm of a product. The logarithm of a product is the sum of their logarithms:

$$\log_a(xy) = \log_a x + \log_a y$$

This means that the calculation of the logarithm of a product, such as 6×8, is the same as the sum of the logarithm of 6 and that of 8. To illustrate,

$$\log_{10} 6 \times 8 = \log_{10} 6 + \log_{10} 8$$

Where is this coming from? From the index law: $a^{x+y} = a^x \times a^y$. The addition in of the indices become the product of the exponentials. In logarithms, the product of the value become the addition of logarithms.

Law 3.3 — Logarithm of a quotient. The logarithm of a quotient is the difference of their logarithms:

$$\log_a \frac{x}{y} = \log_a x - \log_a y \quad .$$

So if you want to calculate the logarithm of something like $4/10$, you just need to do this:

$$\log_{10} \frac{4}{10} = \log_{10} 4 - \log_{10} 10 \quad .$$

This law is coming from the index subtraction law:

$$a^{x-y} = \frac{a^x}{a^y}$$

While indices turn a subtraction to a division, logarithms turn a division into a subtract - they are opposite to each other!

Law 3.4 — Logarithm of a power.

$$\log_a x^y = y \log_a x \quad .$$

This tells us that the logarithm brings the power outside of the logarithm. For example, $\log 2^4 = 4 \times \log 2$; the index 4 was brought before the logarithm.

Law 3.5 — Logarithm of 1 and the base.

$$\log_a 1 = 0 \quad \log_a a = 1 \quad .$$

This law is coming from the well known index properties:

$$a^0 = 1 \quad a^1 = a \quad .$$

The law is very handy. Let use it in combination with the previous law to solve the following problem.

■ **Example 2** Simplify the following expression into a single logarithm without using a calculator:

$$-2\log_2 3 + 2\log_2 4$$

Solution:

$$\begin{aligned}
-2\log_2 3 + 2\log_2 4 &= \log_2 3^{-2} + \log_2 4^2 \\
&= \log_2 3^{-2} \times 4^2 \\
&= \log_2 \frac{16}{9}
\end{aligned}$$

■ **Example 3** Evaluate the following logarithm without using a calculator:

$$2\log_4 \sqrt{2} + l\log_4 \frac{1}{5} + \log_4 160$$

Solution:

$$\begin{aligned}
2\log_4 \sqrt{2} + \log_4 \frac{1}{5} + \log_4 160 &= \log_4 \sqrt{2}^2 \times \frac{1}{5} \times 160 \\
&= \log_4 2 \times \frac{1}{5} \times 160 \\
&= \log_4 64 = 3
\end{aligned}$$

Law 3.6 — Change of logarithm base.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

That's a very important law, because it enables us to use the calculator for evaluating the logarithm with respect to *any* base. Let's say you want to evaluate $\log_5 14$ using the calculator. For that, we use the above law to express all the logarithms in terms of the base-10 logarithm:

$$\log_5 14 = \frac{\log_{10} 14}{\log_{10} 5}$$

4 Exercise

Now let's do some exercise problems on www.mathpractice.xyz.
Exercise 1 Click here: Evaluate logarithms.
Exercise 2 Click here: Simplify logarithmic expressions.