MATHMATICAL METHODS

Written examination: SAC

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
6	**6**	**36**

- Students are to write in blue or black pen.
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book
- Formula sheet
- Working space is provided throughout the book

Instructions

- Write your **student number** in the space provided above on this page
- Unless otherwise indicated, the diagrams in this book are not drawn to scale
- All written responses must be in English

At the end of the examination

- You may keep the formula sheet

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1

(11 marks)

a If $f(x)$ is defined as

$$f(x) = 5x^2 - 5x - 3$$

and $g(x)$ is defined as

$$g(x) = -5x^2 - 5x - 4$$

use the chain rule to find $g^{'}(f(x))$.

A

$$-500x^3 + 745x^2 - 5x - 129$$

B

$$-10x^2 - 1$$

C

$$-500x^3 + 750x^2 - 125$$

D

$$-1000x^3 + 1500x^2 - 250$$

(2 marks)

b Differentiate the following expression with respect to u:

$$sin(sin(u^3 + u^2 + 4u - 3))$$

(2 marks)

c Differentiate the following expression with respect to m

$$\frac{-2m-5}{3m^2+2m-4}$$

(1 mark)

d For the function

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 9x - 1$$

find the two points at which the tangents of the graph of $f(x)$ are perpendicular to the line connecting the two points $(-3, -4)$ and $(1, -\frac{8}{3})$. (3 marks)

e Differentiate the following expression with respect to w

$$\frac{4w-1}{\sqrt{-5w^2 - w - 1}}$$

(1 mark)

f Let

$$f(x) = ax^2 + bx^3 + cx + 3x^4 + 48$$

If $(6, -1680)$, $(-1, 35)$ and $(0, 48)$ are stationary points, find the values of a, b and c.

A

$a = -39, b = -19$ and $c = 0$

B

$a = -36, b = -20$ and $c = 0$

C

$a = -33, b = -20$ and $c = 2$

D

$a = -37, b = -17$ and $c = -1$

(2 marks)

Question 2

(4 marks)

a The volume of water in a tank can be modelled as a function of t minutes since the time the tap was opened using the following formula:

$$-2t^3+3t^2+5t+64$$

Find the average water loss between 10 and 14 minutes to 3 significant figures. (1 mark)

b Let

$$f(x)=\frac{x^3}{3}-\frac{13x^2}{2}+36x-5$$

where

$$x\in[1,6]$$

Find the x coordinate of the local maximum of $f(x)$ within the given domain. (1 mark)

c Let

$$f(x)=ax+2x^3-6x^2+24$$

If $(5,-326)$ is a stationary point, find the value of a. (1 mark)

d A rectangle has perimeter 178 m. If one side has length x m, what is the maximum possible area of the rectangle? (1 mark)

Question 3

The height of a marina pontoon above its reference level is modelled by

$$h(t)=3-\frac{3}{2}\sin\left(\frac{\pi t}{5}\right),$$

where $h(t)$ is measured in metres and t is the number of hours after 6 am.

a Differentiate $h(t)$ to find the rate at which the pontoon height is changing. (1 mark)

b Find the rate of change of the pontoon height at $t=1$ hour and state whether the pontoon is rising or falling. (2 marks)

c Determine the first time after 6 am that the pontoon reaches its lowest point. (2 marks) d During the first 10 hours, when is the pontoon rising most rapidly, and what is that rate? (2 marks)

Question 4

Let $f(x) = 3x + 2$ and $g(x) = -x^2 + 4x + 14$.

a State the domain of f. (1 mark)

b State the range of f. (1 mark)

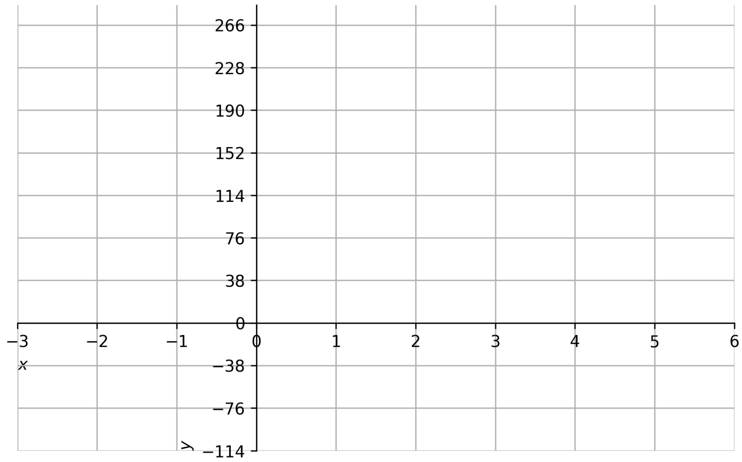
c Find the inverse function $f^{-1}(x)$. (1 mark)

d Solve the equation $f(x) = g(x)$ and hence find the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$. (2 marks)

Question 5

Let $f : R \to R$, $f(x) = x^4 - 3x^3 - 11x^2 + 3x + 10$.

Sketch the graph of $f(x)$ on the following grid.



(4 marks)

Question 6

Consider the functions $f : D \to R$, $f(x) = x^2 + 2x - 3$ and $g : D \to R$, $g(x) = 1 + e^{-x}$.

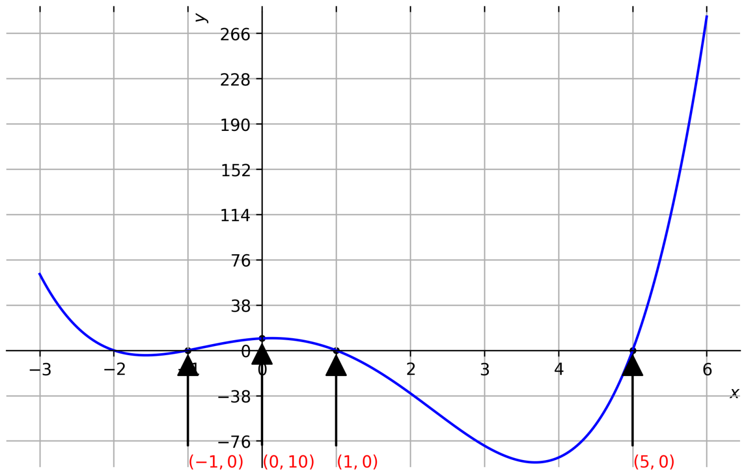
a Differentiate $f(g(x))$ with respect to x. (2 marks)

b Find the maximal domain of $g(f(x))$. (1 mark)

c Find the range of $f(g(x))$. (1 mark)

d Find the range of $g(f(x))$. (1 mark)

Solutions

1. (a) C
   (b)
$$\left(3u^2+2u+4\right)\cos\left(u^3+u^2+4u-3\right)\cos\left(\sin\left(u^3+u^2+4u-3\right)\right)$$
   (c)
$$\frac{6\left(m^2+5m+3\right)}{9m^4+12m^3-20m^2-16m+16}$$
   (d) 2 and -3
   (e)
$$\frac{-40w^2-8w+(4w-1)(10w+1)-8}{2\left(-5w^2-w-1\right)^{\frac{3}{2}}}$$
   (f) B
2. (a) 795
   (b)
$$4$$
   (c) -90
   (d)
$$1980.25$$
3. (a) Differentiate the given function to obtain $h'(t)=-\frac{3\pi\cos\left(\frac{\pi t}{5}\right)}{10}$.
   (b) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $h'(1)=-\frac{3\pi\left(1+\sqrt{5}\right)}{40}\approx -0.762$ m h^{-1}, so the pontoon is falling..
   (c) Set the model equal to the required value and solve for the variable. This gives $t=2.5$ hours.
   (d) The pontoon rises most rapidly at $t=5$ hours, with $h'(5)=\frac{3\pi}{10}\approx 0.942$ m h^{-1}.
4. (a) $\mathbb{R}$
   (b) $(-\infty,\infty)$
   (c) $f^{-1}(x)=\frac{x}{3}-\frac{2}{3}$
   (d) $(-3,-7)$ and $(4,14)$
5. 

6. (a) Use the chain rule and simplify. This gives $2\left(-2e^x-1\right)e^{-2x}$.
   (b) Using the required working, the answer is $\mathbb{R}$.
   (c) Using the required working, the answer is $(0,\infty)$.
   (d) Using the required working, the answer is $\left(1,1+e^4\right]$.