

MATHEMATICAL METHODS

Written examination: SAC

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK Structure of book

<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
6	6	89

- Students are to write in blue or black pen.
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book
- Formula sheet
- Working space is provided throughout the book

Instructions

- Write your **student number** in the space provided above on this page
- Unless otherwise indicated, the diagrams in this book are not drawn to scale
- All written responses must be in English

At the end of the examination

- You may keep the formula sheet

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1**(7 marks)**

- a The following formula expresses the position function of a particle moving on a straight line as a function of t seconds. Find the average speed of the particle between 10 and 15 seconds.

$$-5t^2 - 2t - 3$$

(2 marks)

- b Let

$$f(x) = -\frac{5x^3}{3} - 10x^2 + 300x + 10$$

where

$$x \in [-14, -8]$$

Find the minimum value of $f(x)$ within the given domain. **(2 marks)**

- c Evaluate the derivative of the following polynomial with respect to p , where $p = 5$.

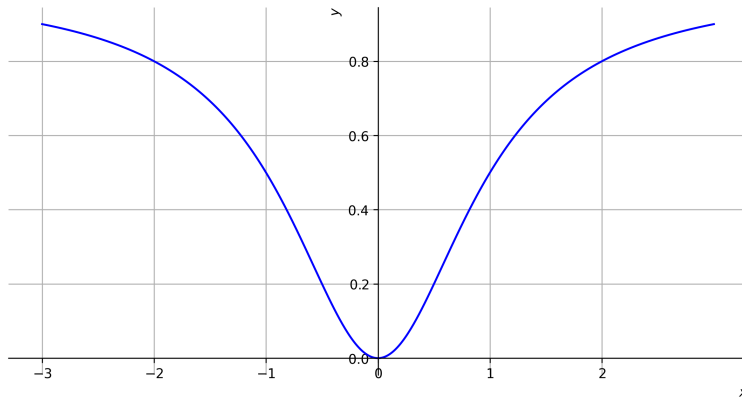
$$4p^2 - p + 2$$

(1 mark)

- d Bilal is designing a display shelf for a museum. The outline of a part of the shelf should look like the curve of the following function:

$$f(x) = \frac{x^2}{x^2 + 1}$$

where x and $f(x)$ are in metres. Find the rate of change of $f(x)$ at $x = 3$ m. Write your answer correct to 3 significant figures. **(2 marks)**

**Question 2****(5 marks)**a If $f(x)$ is defined as

$$f(x) = x^2 - 5x - 3$$

and $g(x)$ is defined as

$$g(x) = -4x^2 + 4x - 1$$

use the chain rule to find $g'(f(x))$.

A

$$-16x^3 + 116x^2 - 140x - 141$$

B

$$-192x^3 + 1440x^2 - 1728x - 1680$$

C

$$-5x^2 + 9x + 2$$

D

$$-16x^3 + 120x^2 - 144x - 140$$

(1 mark)b Differentiate the following expression with respect to l :

$$(l^4 + 3l^3 - 5l^2 - l - 2)^{\frac{4}{7}}$$

A

$$\frac{-4l^3 - 9l^2 + 10l + 1}{21(l^4 + 3l^3 - 5l^2 - l - 2)^{\frac{11}{7}}}$$

B

$$\frac{76(-4l^3 - 9l^2 + 10l + 1)}{7(l^4 + 3l^3 - 5l^2 - l - 2)^{\frac{11}{7}}}$$

C

$$\frac{4(-4l^3 - 9l^2 + 10l + 1)}{7(l^4 + 3l^3 - 5l^2 - l - 2)^{\frac{11}{7}}}$$

D

$$-4l^2 - l + 3$$

(1 mark)

c Differentiate the following expression with respect to a

$$\log(4a^2 + a + 1)$$

A

$$\frac{11(8a+1)}{4a^2+a+1}$$

B

$$-4a^3 - 4a^2 + 2a + 1$$

C

$$\frac{8a+1}{4a^2+a+1}$$

D

$$\frac{8a+1}{8(4a^2+a+1)}$$

(1 mark)

d Find the expression for

$$\frac{dy}{dx}$$

for the following:

$$\frac{9xy}{20} = -\frac{9x^3}{20} - \frac{9x^2}{5} + \frac{9x}{5}$$

A

$$-10x - 20$$

B

$$-2x - 4$$

C

$$-\frac{x}{5} - \frac{2}{5}$$

D

$$-4x^3 + 4x^2 - 4x - 1$$

(1 mark)

e Differentiate the following expression with respect to t

$$-12e^{-5t^2w} \tan^2(4w)$$

A

$$\frac{120twe^{-5t^2w} \tan^2(4w)}{17}$$

B

$$120twe^{-5t^2w} \tan^2(4w)$$

C

$$4e^{6w} \tan(2w^2)$$

D

$$720twe^{-5t^2w} \tan^2(4w)$$

(1 mark)

Question 3

- a A vibration sensor is attached to a reference marker on a laboratory calibration platform. For the first trial, the marker's vertical displacement from its centre line is modelled by

$$y(t) = \frac{3 \cos(5\sqrt{10}t)}{20}.$$

Here t is measured in seconds and y in metres. This trial is called Model A.

- i Differentiate $y(t)$ to find $y'(t)$. (1 mark)

- ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

- iii Determine the maximum speed during the first full period, and state the times at which it occurs. (2 marks)

- iv Differentiate again to find $y''(t)$, then evaluate $y''(0.10)$. (2 marks)

- v State the amplitude and period of Model A. (1 mark)

- b The same calibration platform is then run at a faster setting while keeping the same maximum displacement. The displacement is modelled by

$$y(t) = \frac{3 \cos(20t)}{20}.$$

This trial is called Model B.

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- i State the amplitude and period of Model B. (1 mark)

- ii Express the period of Model B as a percentage of the period of Model A, correct to one decimal place. (2 marks)

- iii Differentiate $y(t)$ to find $y'(t)$, then determine the maximum speed during the first full period and the times at which it occurs. (2 marks)

- iv Find the greatest magnitude of $y''(t)$ for Model B and compare it with the corresponding value for Model A. (2 marks)

- v State the range of Model B. (1 mark)

- c A lighter sensor mount is tested next. The recorded displacement has a smaller maximum displacement and a different period:

$$y(t) = \frac{\cos(8\sqrt{5}t)}{10}.$$

This trial is called Model C.

- i Express the period of Model C as a percentage of the period of Model B, correct to one decimal place. (2 marks)

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- ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

- iii Find the maximum speed and the greatest magnitude of $y''(t)$. (2 marks)

- iv State the amplitude and range of Model C. (1 mark)

- d In a final trial, the marker starts at the centre line and initially moves upward. Its displacement is modelled by

$$y(t) = \frac{\sin(20t)}{10}.$$

This trial is called Model D.

- i Differentiate to obtain expressions for $y'(t)$ and $y''(t)$. (2 marks)

- ii Find the first time after $t = 0$ at which $y(t)$ reaches its maximum value, and state that value. (2 marks)

- iii Find the first positive value of t for which $y(t) = 0$ again, and give the corresponding value of $y'(t)$. (2 marks)

- iv State the amplitude and period of Model D. (1 mark)

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- v Determine the maximum magnitude of $y''(t)$. (1 mark)
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Question 4

- a Ecologists are recording the number of Eastern Quolls living in a fenced sanctuary. Let $P(t)$ be the population, in individuals, t years after the release program began. For the initial stage of the study, the model used is

$$P(t) = 120e^{0.30t}.$$

- i Differentiate the model and show that the instantaneous growth rate is $\frac{dP}{dt} = 36e^{0.30t}$ individuals per year. (1 mark)
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- ii Evaluate the growth rate at $t = 2$ years and interpret that value in context. (2 marks)
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- iii Find the first time at which the growth rate is greater than 150 individuals per year. Give your answer correct to one decimal place. (2 marks)
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- iv Compute the average growth rate over the interval $0 \leq t \leq 5$, and compare it with the instantaneous growth rate at $t = 5$. (2 marks)
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-
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- v Use differentiation to explain why the growth rate is increasing throughout this stage. (1 mark)

- b After 6 years, a feral-cat breach causes an immediate loss of 25% of the Eastern Quolls and the subsequent exponential growth is slower. The revised population model is

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \leq t \leq 6, \\ 0.75(120e^{0.30 \times 6})e^{0.18(t-6)}, & t > 6, \end{cases}$$

where t is still measured in years from the initial release.

- i Simplify the branch for $t > 6$ so that it is written in the form $P(t) = Ae^B e^{0.18(t-6)}$. (1 mark)

- ii Differentiate both branches to obtain a piecewise expression for $\frac{dP}{dt}$. (2 marks)

- iii Find the population size and growth rate just before and just after the breach at $t = 6$. State the percentage change in each quantity. (3 marks)

- iv Measure time from the breach. After how many years will the population recover to the level it had immediately before $t = 6$? Give your answer correct to one decimal place. (2 marks)

- v Suppose the original exponential model had continued with no breach until $t = 10$. Compare that hypothetical population with the actual population at $t = 10$, stating which is larger and by how many individuals. (2 marks)

- c At $t = 9$ years, three years after the feral-cat breach, a restoration program adds 40 Eastern Quolls and improves habitat quality. The population is then described by

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \leq t \leq 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \leq 9, \\ (90e^{2.34} + 40)e^{0.28(t-9)}, & t > 9, \end{cases}$$

with both changes occurring instantaneously at their stated times.

- i Confirm that the final branch can be written as $P(t) = (90e^{2.34} + 40)e^{0.28(t-9)}$ for $t > 9$. (1 mark)

- ii Differentiate the three branches and write down the corresponding piecewise function for $\frac{dP}{dt}$. (2 marks)

- iii Evaluate the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

- iv How many years after the restoration program starts will the population be twice its size at $t = 9^+$? Give your answer correct to one decimal place. (2 marks)

- v If the middle branch had simply continued for all $t > 6$, compare that model with the actual restoration model at $t = 12$. State the population difference and which model gives the larger population. (2 marks)

- d In a different response plan, the managers begin a supplementary-feeding program at $t = 9$ years and simultaneously release 60 Eastern Quolls. The resulting model becomes

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \leq t \leq 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \leq 9, \\ (90e^{2.34} + 60)e^{0.24(t-9)}, & t > 9, \end{cases}$$

where t is measured in years from the beginning of the study.

- i Show that the final branch simplifies to $P(t) = (90e^{2.34} + 60)e^{0.24(t-9)}$ for $t > 9$. (1 mark)

- ii Differentiate each branch to form the corresponding piecewise growth-rate function $\frac{dP}{dt}$. (2 marks)

- iii Determine the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

- iv Taking $t = 9$ as the start of the feeding program, after how many years will the population double from its value at $t = 9^+$? Give your answer correct to one decimal place. (2 marks)

- v Compare the population at $t = 12$ for this feeding model with the population obtained by extending the middle branch with no support. State which population is larger and by how many individuals. (2 marks)

Question 5

A tank is being filled with water and the height of the water (in metres) after t minutes is modelled by the function $h(t) = t^3 - 30t^2 + 225t$.

- a Find the rate at which the water height is changing after 4 minutes. (1 mark)

- b What is the maximum height the water reaches? (1 mark)

- c After how many minutes does the water reaches its maximum? (1 mark)

Question 6

A toy is launched straight up into the air, and its height (in metres) after t seconds is:

$$h(t) = -t^2 + 6t$$

where $t \geq 0$.

- a Find the expression for the toy's velocity. (1 mark)

- b Determine the time the toy reaches maximum height. (1 mark)

- c Confirm the turning point type using the second derivative. (1 mark)

- d Calculate the maximum height. (1 mark)

- e Estimate the landing time when height becomes zero. (1 mark)

Solutions

1. (a)

$$-127.0$$

- (b)

$$-\frac{6970}{3}$$

- (c)

$$39$$

- (d) 0.06

2. (a) D

- (b) C

- (c) C

- (d) B

- (e) B

3. (a) (i) Differentiate the given function to obtain $y'(t) = -\frac{3\sqrt{10}\sin(5\sqrt{10}t)}{4} \text{ m s}^{-1}$.
 (ii) Using the required working, the answer is $t = \frac{\sqrt{10}\pi}{100} \approx 0.0993 \text{ s}$.
 (iii) Using the required working, the answer is maximum speed $= \frac{3\sqrt{10}}{4} \approx 2.37 \text{ m s}^{-1}$,
 at $t = \frac{\sqrt{10}\pi}{100}$ and $t = \frac{3\sqrt{10}\pi}{100}$.
 (iv) Differentiate the given function to obtain $y''(t) = -\frac{75\cos(5\sqrt{10}t)}{2}$, $y''(0.10) = 0.39$.
 (v) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \quad T = \frac{\sqrt{10}\pi}{25} \approx 0.3974 \text{ s}$$

- (b) (i) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \quad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

- (ii) Using the required working, the answer is $T_B = \frac{\pi}{10} \approx 0.314 \text{ s}$, which is 79.1% of the period of Model A..
- (iii) Differentiate the given function to obtain $y'(t) = -3 \sin(20t)$, maximum speed $= 3.00 \text{ m s}^{-1}$ at $t = \frac{\pi}{40}$ and $t = \frac{3\pi}{40}$.
- (iv) Using the required working, the answer is greatest magnitude of $y''(t)$ for Model B $= 60 = 60 \text{ m s}^{-2}$, while for Model A it is $\frac{75}{2} = 37.5 \text{ m s}^{-2}$. So Model B has the larger maximum second derivative..
- (v) Determine the possible output values from the function. The range is $-0.15 \leq y(t) \leq 0.15$.
- (c) (i) Using the required working, the answer is $T = \frac{\sqrt{5}\pi}{20} \approx 0.3512 \text{ s}$, which is 111.8% of the period of Model B..
- (ii) Using the required working, the answer is $t = \frac{\sqrt{5}\pi}{80} \approx 0.0878 \text{ s}$.
- (iii) Using the required working, the answer is maximum speed $= \frac{4\sqrt{5}}{5} \approx 1.79 \text{ m s}^{-1}$, greatest magnitude of $y''(t) = 32 \approx 32 \text{ m s}^{-2}$.
- (iv) Using the required working, the answer is Amplitude

$$= 0.10 \text{ m}, \quad -0.10 \leq y(t) \leq 0.10$$

- (d) (i) Differentiate the given function to obtain

$$y'(t) = 2 \cos(20t), \quad y''(t) = -40 \sin(20t)$$

- (ii) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives

$$t = \frac{\pi}{40} \approx 0.0785 \text{ s}, \quad y_{\max} = \frac{1}{10} \text{ m}$$

- (iii) Using the required working, the answer is

$$t = \frac{\pi}{20} \approx 0.1571 \text{ s}, \quad y'(t) = -2 \text{ m s}^{-1}$$

- (iv) Using the required working, the answer is Amplitude

$$= 0.1 \text{ m}, \quad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

- (v) Using the required working, the answer is maximum magnitude of $y''(t) = 40.0 = 40 \text{ m s}^{-2}$.

4. (a) (i) Differentiate the given function to obtain $36e^{\frac{3t}{10}}$.
- (ii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives At $t = 2$, $\frac{dP}{dt} = 65.6$ individuals per year, so the quoll population is increasing at this instantaneous rate after 2 years..
- (iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 4.8.

- (iv) Average growth rate = 83.56, instantaneous rate at $t = 5$ is 161.34. The instantaneous rate at $t = 5$ is larger.
- (v) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $\frac{54e^{\frac{3t}{10}}}{5} > 0$, so $\frac{dP}{dt}$ is increasing for all t .
- (b) (i) Using the required working, the answer is $P(t) = 90e^{1.8}e^{0.18(t-6)}$.
- (ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \leq t \leq 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & t > 6. \end{cases}$$

- (iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives Population: $P(6^-) = 725.96$, $P(6^+) = 544.47$ so there is a 25.0% decrease. Growth rate: $P'(6^-) = 217.79$, $P'(6^+) = 98$ so there is a 55.0% decrease..
- (iv) Set the model equal to the required value and solve for the variable. This gives 1.6.
- (v) Set the model equal to the required value and solve for the variable. This gives Without the incursion there would be 1291.69 more quolls at $t = 10$, so the actual model has fewer quolls..
- (c) (i) Using the required working, the answer is $P(t) = (90e^{2.34} + 40)e^{0.28(t-9)}$.
- (ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \leq t \leq 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & 6 < t \leq 9 \\ \frac{7\left(40+90e^{\frac{117}{50}}\right)e^{\frac{7t}{25}-\frac{63}{25}}}{25}, & t > 9. \end{cases}$$

- (iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(9^-) = 934.31$, $P(9^+) = 974.31$, an increase of 4.28%. Growth rate: $P'(9^-) = 168.18$, $P'(9^+) = 272.81$, an increase of 62.22%.
- (iv) Using the required working, the answer is 2.5.
- (v) The actual model has 653.58 more quolls at $t = 12$ than the no-intervention model.
- (d) (i) Substitute the matching value from the previous stage and simplify the expression. This gives $P(t) = (90e^{2.34} + 60)e^{0.24(t-9)}$.
- (ii) Differentiate each branch separately on its stated interval. This gives

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \leq t \leq 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & 6 < t \leq 9 \\ \frac{6\left(60+90e^{\frac{117}{50}}\right)e^{\frac{6t}{25}-\frac{54}{25}}}{25}, & t > 9. \end{cases}$$

- (iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(9^-) = 934.31$, $P(9^+) = 994.31$, an increase of 6.42%. Growth rate: $P'(9^-) = 168.18$, $P'(9^+) = 238.63$, an increase of 41.9%.
- (iv) Using the required working, the answer is 2.9.
- (v) The supplementary-feeding model gives 439.46 more quolls at $t = 12$ years.
5. (a) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 33.

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- (b) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 500.
- (c) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 5.
6. (a) Using the required working, the answer is $6 - 2t$.
- (b) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 3.
- (c) Differentiate the first derivative once more to obtain Maximum since $h''(t) = -2 < 0$.
- (d) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 9.
- (e) Set the model equal to the required value and solve for the variable. This gives 6.