MATHEMATICS ADVANCED

Written examination: School-based Assessment

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
7	**7**	**74**

- Calculators approved by NESA may be used.
- A reference sheet is provided at the back of this paper.
- Write your Centre Number and Student Number at the top of this page.

Question 1

(11 marks)

a If $f(x)$ is defined as

$$f(x) = 4x^2 - 3x + 2$$

and $g(x)$ is defined as

$$g(x) = 5x^2 - 3x - 3$$

use the chain rule to find $g'(f(x))$.

A

$$x^2 - 5$$

B

$$4480x^3 - 5040x^2 + 3164x - 714$$

C

$$320x^3 - 355x^2 + 223x - 54$$

D

$$320x^3 - 360x^2 + 226x - 51$$

(2 marks)

b Differentiate the following expression with respect to m:

$$sin(cos(m^3 - 4m^2 + 3m + 5))$$

(2 marks)

c Differentiate the following expression with respect to y

$$\frac{y+4}{-2y^2 + 4y + 5}$$

(1 mark)

d For the function

$$f(x)=\frac{x^3}{3}+\frac{7x^2}{2}+13x-1$$

find the two points at which the tangents of the graph of $f(x)$ are perpendicular to the line connecting the two points $(2,4)$ and $(6,\frac{8}{3})$. (3 marks)

e Differentiate the following expression with respect to m

$$\frac{2(1-2m)}{\sqrt{4m^2+5m-4}}$$

(1 mark)

f Let

$$f(x)=ax^4+bx^3+c-24x^2+48x$$

If $(2,4)$, $(1,11)$ and $(-2,-124)$ are stationary points, find the values of a, b and c.

A

$$a=1,b=-4andc=-13$$

B

$$a=3,b=-4andc=-12$$

C

$$a=2,b=-6andc=-14$$

D

$$a=3,b=-1andc=-12$$

(2 marks)

Question 2

(4 marks)

a The volume of water in a tank can be modelled as a function of t minutes since the time the tap was opened using the following formula:

$$-9t^3 - 2t^2 - 4t + 23$$

Find the average water loss between 10 and 19 minutes Write your answer to the nearest integer. (1 mark)

b Let

$$f(x) = \frac{2x^3}{3} - 11x^2 + 56x - 8$$

where

$$x \in [3, 6]$$

Find the x coordinate of the local maximum of $f(x)$ within the given domain. (1 mark)

c Let

$$f(x) = ax^3 - 18x^2 + 48x - 30$$

If $(2, 10)$ is a stationary point, find the value of a. (1 mark)

d A rectangular pen is built against a wall using 115 m of fencing for three sides. If the side perpendicular to the wall has length x m, what is the maximum possible area of the pen? (1 mark)

Question 3

a Field scientists at a conservation park are modelling the Eastern Quoll population. If $P(t)$ is the number of quolls t years after the project begins, the initial model is

$$P(t) = 120e^{0.30t}.$$

i Differentiate the model to show that the population is growing at $\frac{dP}{dt} = 36e^{0.30t}$ individuals per year. (1 mark)

ii Find the instantaneous rate of growth after 2 years, and interpret the result.(2 marks)

iii When does the growth rate first become larger than 150 individuals per year? Give your answer correct to one decimal place. (2 marks) iv Determine the mean rate of growth from $t = 0$ to $t = 5$, and compare it with the instantaneous rate at $t = 5$. (2 marks)

v Explain, using differentiation, why the growth rate is increasing throughout the time shown. (1 mark)

b After 6 years, a sudden predator incursion removes 25% of the Eastern Quolls and the subsequent exponential growth is less rapid. The new model is

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \le t \le 6, \\ 0.75\left(120e^{0.30\times 6}\right)e^{0.18(t-6)}, & t > 6, \end{cases}$$

where t is still measured from the beginning of the project.

i Show that the second branch simplifies to $P(t) = 90e^{1.8}e^{0.18(t-6)}$ for $t > 6$. (1 mark)

ii Differentiate the two branches and hence obtain a piecewise formula for $\frac{dP}{dt}$.(2 marks)

iii Calculate the population and growth rate immediately before and immediately after $t = 6$, and state the percentage decrease in each. (3 marks)

iv How long after the incursion does the population return to the size it had at $t = 6^{-}$? Give your answer correct to one decimal place. (2 marks) v Compare the actual population at $t = 10$ with the population that would have occurred if the original exponential rule had continued unchanged. State which is larger and by how many individuals. (2 marks)

c At $t = 9$ years, the park completes habitat restoration and releases 40 extra Eastern Quolls. From this point, the population is modelled by

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \le t \le 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \le 9, \\ (90e^{2.34}+40)e^{0.28(t-9)}, & t > 9, \end{cases}$$

where the changes at $t = 6$ and $t = 9$ are instantaneous.

i Verify that the final branch may be written as $P(t) = (90e^{2.34}+40)e^{0.28(t-9)}$. (1 mark)

ii Differentiate each branch and write down the piecewise growth-rate function $\frac{dP}{dt}$.(2 marks)

iii Find the population and growth rate immediately before and after $t = 9$, and state the percentage increase in each. (3 marks)

iv How many years after the restoration release will the population double from its value at $t = 9^+$? Give your answer correct to one decimal place. (2 marks)

v Compare the actual model at $t = 12$ with the model obtained by extending the middle branch past $t = 9$ with no restoration. State which population is larger and by how many individuals. (2 marks) d In another intervention plan, the park starts a supplementary-feeding program at $t = 9$ years and releases 60 Eastern Quolls. The corresponding population model is

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \le t \le 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \le 9, \\ (90e^{2.34} + 60)e^{0.24(t-9)}, & t > 9, \end{cases}$$

where t is measured from the beginning of the study.

i Confirm that the final branch simplifies to $P(t) = (90e^{2.34} + 60)e^{0.24(t-9)}$. (1 mark)

ii Differentiate all three branches to obtain the piecewise formula for $\frac{dP}{dt}$. (2 marks)

iii Calculate the population and growth rate immediately before and after $t = 9$, and state the percentage increase in each quantity. (3 marks)

iv How many years after $t = 9$ will the population double from its value immediately after the support program begins? Give your answer correct to one decimal place.(2 marks)

v At $t = 12$, compare this support model with the population obtained by extending the middle branch without intervention. State which population is larger and by how many individuals. (2 marks)

Question 4

The height of a marina pontoon above its reference level is modelled by

$$h(t) = 3 - \frac{3}{2}\sin\left(\frac{\pi t}{5}\right),$$

where $h(t)$ is measured in metres and t is the number of hours after 6 am.

a Differentiate $h(t)$ to find the rate at which the pontoon height is changing. (1 mark)

b Find the rate of change of the pontoon height at $t = 1$ hour and state whether the pontoon is rising or falling. (2 marks)

c Determine the first time after 6 am that the pontoon reaches its lowest point. (2 marks)

d During the first 10 hours, when is the pontoon rising most rapidly, and what is that rate? (2 marks)

Question 5

Let $f(x) = x^3 + 3x^2 + 3x - 2$ and $g(x) = 12x - 7$.

a State the domain of f. (1 mark)

b State the range of f. (1 mark)

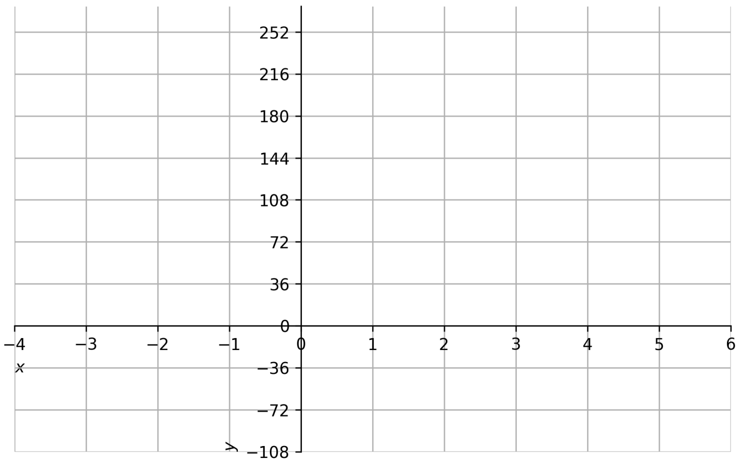
c Find the inverse function $f^{-1}(x)$. (1 mark)

d Solve the equation $f(x) = g(x)$ and hence find the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$. (2 marks)

Question 6

Let $f : R \to R$, $f(x) = x^4 - 5x^3 - 7x^2 + 41x - 30$.

Sketch the graph of $f(x)$ on the following grid.



(4 marks)

Question 7

Consider the functions $f : D \to R$, $f(x) = 12x^2 - 10x + 2$ and $g : D \to R$, $g(x) = e^{3x}$.

a Differentiate $f(g(x))$ with respect to x. (2 marks) b Find the maximal domain of $g(f(x))$. (1 mark)

c Find the range of $f(g(x))$. (1 mark)

d Find the range of $g(f(x))$. (1 mark)

Solutions

1. (a) D
   (b)
   $$\left(-3m^2+8m-3\right)\sin\left(m^3-4m^2+3m+5\right)\cos\left(\cos\left(m^3-4m^2+3m+5\right)\right)$$
   (c)
   $$\frac{-2y^2+4y+4\left(y-1\right)\left(y+4\right)+5}{\left(-2y^2+4y+5\right)^2}$$
   (d) -5 and -2
   (e)
   $$\frac{11-18m}{\left(4m^2+5m-4\right)^{\frac{3}{2}}}$$
   (f) B
2. (a) 5921
   (b)
   $$4$$
   (c) 2
   (d)
   $$\frac{13225}{8}$$

3. (a) (i) Differentiate the given function to obtain $36e^{\frac{3t}{10}}$.
   (ii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives At $t = 2,\ \frac{dP}{dt} = 65.6$ individuals per year, so the quoll population is increasing at this instantaneous rate after 2 years..
   (iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 4.8.
   (iv) Average growth rate$= 83.56$, instantaneous rate at $t = 5$ is 161.34. The instantaneous rate at $t = 5$ is larger.
   (v) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $\frac{54e^{\frac{3t}{10}}}{5} > 0$, so $\frac{dP}{dt}$ is increasing for all t.

   (b) (i) Substitute the matching value from the previous stage and simplify the expression. This gives $P(t) = 90e^{1.8}e^{0.18(t-6)}$.
   (ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \le t \le 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & t > 6. \end{cases}$$

.
   (iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(6^-) = 725.96,\ P(6^+) = 544.47$ so there is a 25.0% decrease. Growth rate: $P'(6^-) = 217.79,\ P'(6^+) = 98$ so there is a 55.0% decrease..
   (iv) Set the model equal to the required value and solve for the variable. This gives 1.6.
   (v) Using the required working, the answer is Without the incursion there would be 1291.69 more quolls at $t = 10$, so the actual model has fewer quolls..

   (c) (i) Using the required working, the answer is $P(t) = (90e^{2.34} + 40)e^{0.28(t-9)}$.
   (ii) Differentiate each branch separately on its stated interval. This gives

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \le t \le 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & 6 < t \le 9 \\ \frac{7\left(40+90e^{\frac{117}{50}}\right)e^{\frac{7t}{25}-\frac{63}{25}}}{25}, & t > 9. \end{cases}$$

.
   (iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives Population: $P(9^-) = 934.31,\ P(9^+) = 974.31$, an increase of 4.28%. Growth rate: $P'(9^-) = 168.18,\ P'(9^+) = 272.81$, an increase of 62.22%.
   (iv) Using the required working, the answer is 2.5.
   (v) The actual model has 653.58 more quolls at $t = 12$ than the no-intervention model.

   (d) (i) Using the required working, the answer is $P(t) = (90e^{2.34} + 60)e^{0.24(t-9)}$.
   (ii) Differentiate the given function to obtain

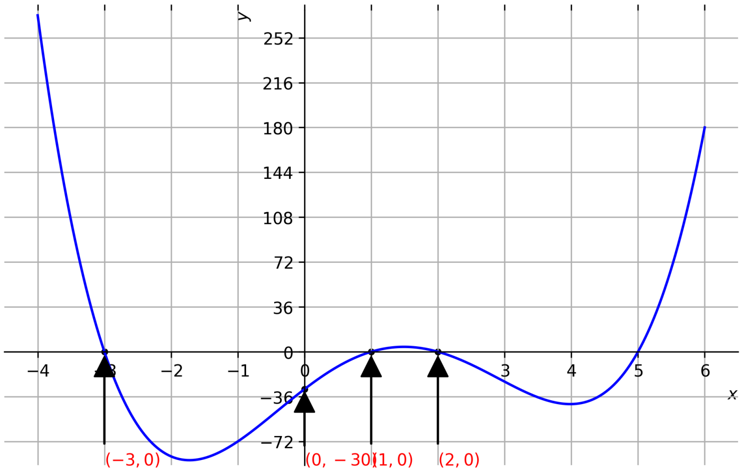
$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \le t \le 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & 6 < t \le 9 \\ \frac{6\left(60+90e^{\frac{117}{50}}\right)e^{\frac{6t}{25}-\frac{54}{25}}}{25}, & t > 9. \end{cases}$$

.
   (iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives Population: $P(9^-) = 934.31,\ P(9^+) = 994.31$, an increase of 6.42%. Growth rate: $P'(9^-) = 168.18,\ P'(9^+) = 238.63$, an increase of 41.9%.
   (iv) Using the required working, the answer is 2.9.

(v) The supplementary-feeding model gives 439.46 more quolls at $t = 12$ years.

4. (a) Differentiate the given function to obtain $h'(t) = -\frac{3\pi \cos\left(\frac{\pi t}{5}\right)}{10}$.
   (b) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $h'(1) = -\frac{3\pi\left(1+\sqrt{5}\right)}{40} \approx -0.762$ m h^{-1}, so the pontoon is falling..
   (c) Set the model equal to the required value and solve for the variable. This gives $t = 2.5$ hours.
   (d) The pontoon rises most rapidly at $t = 5$ hours, with $h'(5) = \frac{3\pi}{10} \approx 0.942$ m h^{-1}.

5. (a) $\mathbb{R}$
   (b) $(-\infty, \infty)$
   (c) $f^{-1}(x) = \sqrt[3]{x+3} - 1$
   (d) $(-5, -67)$ and $(1, 5)$

6. 

7. (a) Use the chain rule and simplify. This gives $\left(72e^{3x} - 30\right) e^{3x}$.
   (b) Using the required working, the answer is $\mathbb{R}$.
   (c) Using the required working, the answer is $\left[-\frac{1}{12}, \infty\right)$.
   (d) Using the required working, the answer is $\left[e^{-\frac{1}{4}}, \infty\right)$.