MATHEMATICS ADVANCED

Written examination: School-based Assessment

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
6	**6**	**89**

- Calculators approved by NESA may be used.
- A reference sheet is provided at the back of this paper.
- Write your Centre Number and Student Number at the top of this page.

Question 1

(7 marks)

a The following formula expresses the position function of a particle moving on a straight line as a function of t seconds. Find the average speed of the particle between 10 and 12 seconds.

$$-t^2+3t+3$$

(2 marks)

b Let

$$f(x)=\frac{4x^3}{3}+10x^2-200x-8$$

where

$$x\in[-14,-5]$$

Find the minimum value of $f(x)$ within the given domain. (2 marks)

c Evaluate the derivative of the following polynomial with respect to b, where b = -4.

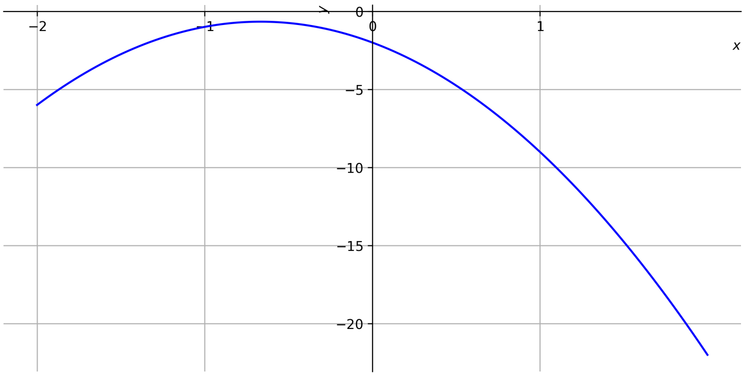
$$-2b^2+2b-1$$

(1 mark)

d Cody is designing a display shelf for a museum. The outline of a part of the shelf should look like the curve of the following function:

$$f(x)=-3x^2-4x-2$$

where x and $f(x)$ are in metres. Find the rate of change of $f(x)$ at $x=-2$ m. Write your answer correct to 3 significant figures. (2 marks)



Question 2

(5 marks)

a If $f(x)$ is defined as

$$f(x) = 3x^2 - x - 3$$

and $g(x)$ is defined as

$$g(x) = -5x^2 - 2x + 4$$

use the chain rule to find $g'(f(x))$.

A

$$-3240x^3 + 1620x^2 + 2844x - 504$$

B

$$-180x^3 + 85x^2 + 156x - 24$$

C

$$-8x^2 - x + 7$$

D

$$-180x^3 + 90x^2 + 158x - 28$$

(1 mark)

b Differentiate the following expression with respect to r:

$$(5r-5)^{\frac{4}{3}}$$

A

$$-\frac{4 \times 5^{\frac{2}{3}}}{525\,(r-1)^{\frac{7}{3}}}$$

B

$$-r^2 - r - 4$$

C

$$-\frac{4 \times 5^{\frac{2}{3}}}{15\,(r-1)^{\frac{7}{3}}}$$

D

$$-\frac{4 \times 5^{\frac{2}{3}}}{75\,(r-1)^{\frac{7}{3}}}$$

(1 mark)

c Differentiate the following expression with respect to u

$$-2\log\left(-4u^2+5u+5\right)$$

A

$$-u^3+3u^2+3u-2$$

B

$$\frac{2(8u-5)}{-4u^2+5u+5}$$

C

$$\frac{8u-5}{6(-4u^2+5u+5)}$$

D

$$\frac{26(8u-5)}{-4u^2+5u+5}$$

(1 mark)

d Find the expression for

$$\frac{dy}{dx}$$

for the following:

$$0=0$$

A

$$2x-5$$

B

$$2x^3-3x^2-2x-3$$

C

$$\frac{x}{5}-\frac{1}{2}$$

D

$$10x-25$$

(1 mark)

e Differentiate the following expression with respect to n

$$4e^{5b^2n}\tan\left(b^2n^2\right)$$

A

$$-8e^{5b^2}\sin^2\left(bn^2\right)$$

B

$$8b^2ne^{5b^2n}\tan^2\left(b^2n^2\right)+8b^2ne^{5b^2n}+20b^2e^{5b^2n}\tan\left(b^2n^2\right)$$

C

$$96b^2ne^{5b^2n}\tan^2\left(b^2n^2\right)+96b^2ne^{5b^2n}+240b^2e^{5b^2n}\tan\left(b^2n^2\right)$$

D

$$\frac{8b^2ne^{5b^2n}\tan^2\left(b^2n^2\right)}{19}+\frac{8b^2ne^{5b^2n}}{19}+\frac{20b^2e^{5b^2n}\tan\left(b^2n^2\right)}{19}$$

(1 mark)

Question 3

a A vibration sensor is attached to a reference marker on a laboratory calibration platform. For the first trial, the marker's vertical displacement from its centre line is modelled by

$$y(t) = \frac{3\cos\left(5\sqrt{10}t\right)}{20}.$$

Here t is measured in seconds and y in metres. This trial is called Model A.

i Differentiate $y(t)$ to find $y'(t)$. (1 mark)

ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Determine the maximum speed during the first full period, and state the times at which it occurs. (2 marks)

iv Differentiate again to find $y''(t)$, then evaluate $y''(0.10)$. (2 marks)

v State the amplitude and period of Model A. (1 mark)

b The same calibration platform is then run at a faster setting while keeping the same maximum displacement. The displacement is modelled by

$$y(t) = \frac{3\cos(20t)}{20}.$$

This trial is called Model B.

i State the amplitude and period of Model B. (1 mark)

ii Express the period of Model B as a percentage of the period of Model A, correct to one decimal place. (2 marks)

iii Differentiate $y(t)$ to find $y'(t)$, then determine the maximum speed during the first full period and the times at which it occurs. (2 marks)

iv Find the greatest magnitude of $y''(t)$ for Model B and compare it with the corresponding value for Model A. (2 marks)

v State the range of Model B. (1 mark)

c A lighter sensor mount is tested next. The recorded displacement has a smaller maximum displacement and a different period:

$$y(t) = \frac{\cos\left(8\sqrt{5}t\right)}{10}.$$

This trial is called Model C.

i Express the period of Model C as a percentage of the period of Model B, correct to one decimal place. (2 marks) ii Find the first value of $t > 0$ for which $y(t) = 0$. (2 marks)

iii Find the maximum speed and the greatest magnitude of $y''(t)$. (2 marks)

iv State the amplitude and range of Model C. (1 mark)

d In a final trial, the marker starts at the centre line and initially moves upward. Its displacement is modelled by

$$y(t) = \frac{\sin(20t)}{10}.$$

This trial is called Model D.

i Differentiate to obtain expressions for $y'(t)$ and $y''(t)$. (2 marks)

ii Find the first time after $t = 0$ at which $y(t)$ reaches its maximum value, and state that value. (2 marks)

iii Find the first positive value of t for which $y(t) = 0$ again, and give the corresponding value of $y'(t)$. (2 marks)

iv State the amplitude and period of Model D. (1 mark) v Determine the maximum magnitude of $y''(t)$. (1 mark)

Question 4

a Ecologists are recording the number of Eastern Quolls living in a fenced sanctuary. Let $P(t)$ be the population, in individuals, t years after the release program began. For the initial stage of the study, the model used is

$$P(t) = 120e^{0.30t}.$$

i Differentiate the model and show that the instantaneous growth rate is $\frac{dP}{dt} = 36e^{0.30t}$ individuals per year. (1 mark)

ii Evaluate the growth rate at $t = 2$ years and interpret that value in context. (2 marks)

iii Find the first time at which the growth rate is greater than 150 individuals per year. Give your answer correct to one decimal place. (2 marks)

iv Compute the average growth rate over the interval $0 \leq t \leq 5$, and compare it with the instantaneous growth rate at $t = 5$. (2 marks)

v Use differentiation to explain why the growth rate is increasing throughout this stage. (1 mark) b After 6 years, a feral-cat breach causes an immediate loss of 25% of the Eastern Quolls and the subsequent exponential growth is slower. The revised population model is

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \le t \le 6, \\ 0.75\left(120e^{0.30\times 6}\right)e^{0.18(t-6)}, & t > 6, \end{cases}$$

where t is still measured in years from the initial release.

i Simplify the branch for $t > 6$ so that it is written in the form $P(t) = Ae^{B}e^{0.18(t-6)}$.(1 mark)

ii Differentiate both branches to obtain a piecewise expression for $\frac{dP}{dt}$. (2 marks)

iii Find the population size and growth rate just before and just after the breach at $t = 6$. State the percentage change in each quantity. (3 marks)

iv Measure time from the breach. After how many years will the population recover to the level it had immediately before $t = 6$? Give your answer correct to one decimal place. (2 marks)

v Suppose the original exponential model had continued with no breach until $t = 10$. Compare that hypothetical population with the actual population at $t = 10$, stating which is larger and by how many individuals. (2 marks) c At $t = 9$ years, three years after the feral-cat breach, a restoration program adds 40 Eastern Quolls and improves habitat quality. The population is then described by

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \leq t \leq 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \leq 9, \\ (90e^{2.34} + 40)e^{0.28(t-9)}, & t > 9, \end{cases}$$

with both changes occurring instantaneously at their stated times.

i Confirm that the final branch can be written as $P(t) = (90e^{2.34} + 40)e^{0.28(t-9)}$ for $t > 9$. (1 mark)

ii Differentiate the three branches and write down the corresponding piecewise function for $\frac{dP}{dt}$. (2 marks)

iii Evaluate the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

iv How many years after the restoration program starts will the population be twice its size at $t = 9^+$? Give your answer correct to one decimal place. (2 marks)

v If the middle branch had simply continued for all $t > 6$, compare that model with the actual restoration model at $t = 12$. State the population difference and which model gives the larger population. (2 marks) d In a different response plan, the managers begin a supplementary-feeding program at $t = 9$ years and simultaneously release 60 Eastern Quolls. The resulting model becomes

$$P(t) = \begin{cases} 120e^{0.30t}, & 0 \leq t \leq 6, \\ 90e^{1.8}e^{0.18(t-6)}, & 6 < t \leq 9, \\ (90e^{2.34} + 60)e^{0.24(t-9)}, & t > 9, \end{cases}$$

where t is measured in years from the beginning of the study.

i Show that the final branch simplifies to $P(t) = (90e^{2.34} + 60)e^{0.24(t-9)}$ for $t > 9$.(1 mark)

ii Differentiate each branch to form the corresponding piecewise growth-rate function $\frac{dP}{dt}$. (2 marks)

iii Determine the population and the growth rate immediately before and immediately after $t = 9$. State the percentage increase in each quantity. (3 marks)

iv Taking $t = 9$ as the start of the feeding program, after how many years will the population double from its value at $t = 9^{+}$? Give your answer correct to one decimal place. (2 marks)

v Compare the population at $t = 12$ for this feeding model with the population obtained by extending the middle branch with no support. State which population is larger and by how many individuals. (2 marks)

Question 5

A tank is being filled with water and the height of the water (in metres) after t minutes is modelled by the function $h(t) = t^3 - 26t^2 + 169t$.

a Find the rate at which the water height is changing after 4 minutes. (1 mark)

b What is the maximum height the water reaches? (1 mark)

c After how many minutes does the water reaches its maximum? (1 mark)

Question 6

A fireworks shell follows a vertical path whose height above the launch point is modelled by

$$h(t) = t^3 - 22t^2 + 121t$$

where $t \geq 0$ is measured in seconds.

a Find the velocity function $h'(t)$. (1 mark)

b Determine the time when the shell reaches its highest point. (1 mark) c Use the second derivative to classify this turning point. (1 mark)

d Find the maximum height reached by the shell. (1 mark)

e Determine when the shell returns to ground level. (1 mark)

Solutions

1. (a)

$$-19.0$$

(b)

$$\frac{3226}{3}$$

(c)

$$18$$

(d) 8

2. (a) D
   (b) D
   (c) B
   (d) A
   (e) B

3. (a) (i) Differentiate the given function to obtain $y'(t) = -\frac{3\sqrt{10}\sin\left(5\sqrt{10}t\right)}{4}$ m s^{-1}.
   (ii) Using the required working, the answer is $t = \frac{\sqrt{10}\pi}{100} \approx 0.0993$ s.
   (iii) Using the required working, the answer is maximum speed $= \frac{3\sqrt{10}}{4} \approx 2.37$ m s^{-1}, at $t = \frac{\sqrt{10}\pi}{100}$ and $t = \frac{3\sqrt{10}\pi}{100}$.
   (iv) Differentiate the given function to obtain $y''(t) = -\frac{75\cos\left(5\sqrt{10}t\right)}{2}, \quad y''(0.10) = 0.39.$
   (v) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\sqrt{10}\pi}{25} \approx 0.3974 \text{ s}$$

.

(b) (i) Using the required working, the answer is Amplitude

$$= 0.15 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(ii) Using the required working, the answer is $T_B = \frac{\pi}{10} \approx 0.314$ s, which is 79.1% of the period of Model A..

(iii) Differentiate the given function to obtain $y'(t) = -3\sin(20t)$, maximum speed $= 3.00$ m s^{-1} at $t = \frac{\pi}{40}$ and $t = \frac{3\pi}{40}$.

(iv) Using the required working, the answer is greatest magnitude of $y''(t)$ for Model B $= 60 = 60$ m s^{-2}, while for Model A it is $\frac{75}{2} = 37.5$ m s^{-2}. So Model B has the larger maximum second derivative..

(v) Determine the possible output values from the function. The range is $-0.15 \le y(t) \le 0.15$.

(c) (i) Using the required working, the answer is $T = \frac{\sqrt{5}\pi}{20} \approx 0.3512$ s, which is 111.8% of the period of Model B..

(ii) Using the required working, the answer is $t = \frac{\sqrt{5}\pi}{80} \approx 0.0878$ s.

(iii) Using the required working, the answer is maximum speed $= \frac{4\sqrt{5}}{5} \approx 1.79$ m s^{-1}, greatest magnitude of $y''(t) = 32 \approx 32$ m s^{-2}.

(iv) Using the required working, the answer is Amplitude

$$= 0.10 \text{ m}, \qquad -0.10 \le y(t) \le 0.10$$

.

(d) (i) Differentiate the given function to obtain

$$y'(t) = 2\cos(20t), \qquad y''(t) = -40\sin(20t)$$

.

(ii) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives

$$t = \frac{\pi}{40} \approx 0.0785 \text{ s}, \qquad y_{\max} = \frac{1}{10} \text{ m}$$

.

(iii) Using the required working, the answer is

$$t = \frac{\pi}{20} \approx 0.1571 \text{ s}, \qquad y'(t) = -2 \text{ m s}^{-1}$$

.

(iv) Using the required working, the answer is Amplitude

$$= 0.1 \text{ m}, \qquad T = \frac{\pi}{10} \approx 0.314 \text{ s}$$

.

(v) Using the required working, the answer is maximum magnitude of $y''(t) = 40.0 = 40$ m s^{-2}.

4. (a) (i) Differentiate the given function to obtain $36e^{\frac{3t}{10}}$.

(ii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives At $t = 2$, $\frac{dP}{dt} = 65.6$ individuals per year, so the quoll population is increasing at this instantaneous rate after 2 years..

(iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 4.8.

(iv) Average growth rate$= 83.56$, instantaneous rate at $t = 5$ is 161.34. The instantaneous rate at $t = 5$ is larger.

(v) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives $\frac{54e^{\frac{3t}{10}}}{5} > 0$, so $\frac{dP}{dt}$ is increasing for all t.

(b) (i) Using the required working, the answer is $P(t) = 90e^{1.8}e^{0.18(t-6)}$.

(ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \le t \le 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & t > 6. \end{cases}$$

.

(iii) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives Population: $P(6^-) = 725.96,\ P(6^+) = 544.47$ so there is a 25.0% decrease. Growth rate: $P'(6^-) = 217.79,\ P'(6^+) = 98$ so there is a 55.0% decrease..

(iv) Set the model equal to the required value and solve for the variable. This gives 1.6.

(v) Set the model equal to the required value and solve for the variable. This gives Without the incursion there would be 1291.69 more quolls at $t = 10$, so the actual model has fewer quolls..

(c) (i) Using the required working, the answer is $P(t) = (90e^{2.34} + 40)e^{0.28(t-9)}$.

(ii) Differentiate the given function to obtain

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \le t \le 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & 6 < t \le 9 \\ \frac{7\left(40+90e^{\frac{117}{50}}\right)e^{\frac{7t}{25}-\frac{63}{25}}}{25}, & t > 9. \end{cases}$$

.

(iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(9^-) = 934.31,\ P(9^+) = 974.31$, an increase of 4.28%. Growth rate: $P'(9^-) = 168.18,\ P'(9^+) = 272.81$, an increase of 62.22%.

(iv) Using the required working, the answer is 2.5.

(v) The actual model has 653.58 more quolls at $t = 12$ than the no-intervention model.

(d) (i) Substitute the matching value from the previous stage and simplify the expression. This gives $P(t) = (90e^{2.34} + 60)e^{0.24(t-9)}$.

(ii) Differentiate each branch separately on its stated interval. This gives

$$\frac{dP}{dt} = \begin{cases} 36e^{\frac{3t}{10}}, & 0 \le t \le 6 \\ \frac{81e^{\frac{9}{5}}e^{\frac{9t}{50}-\frac{27}{25}}}{5}, & 6 < t \le 9 \\ \frac{6\left(60+90e^{\frac{117}{50}}\right)e^{\frac{6t}{25}-\frac{54}{25}}}{25}, & t > 9. \end{cases}$$

.

(iii) Evaluate the relevant branch, and its derivative if required, on both sides of the changeover time. This gives Population: $P(9^-) = 934.31,\ P(9^+) = 994.31$, an increase of 6.42%. Growth rate: $P'(9^-) = 168.18,\ P'(9^+) = 238.63$, an increase of 41.9%.

(iv) Using the required working, the answer is 2.9.

(v) The supplementary-feeding model gives 439.46 more quolls at $t = 12$ years.

5. (a) Differentiate first to get the relevant rate function, then substitute the required value if needed. This gives 9.

(b) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 325.482.

(c) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives $\frac{13}{3}$.

6. (a) Differentiate the given function to obtain $t\left(2t-22\right)+\left(t-11\right)^{2}$.

   (b) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives $\frac{11}{3}$.

   (c) Differentiate the first derivative once more to obtain Maximum since $h''\left(\frac{11}{3}\right)=-22<0$.

   (d) Find the turning time by setting the derivative equal to 0, then substitute back into the original function. This gives 197.185.

   (e) Set the model equal to the required value and solve for the variable. This gives 11.